A DIFFERENTIAL $\mathcal{U}$-MODULE ALGEBRA FOR $\mathcal{U} = \overline{\mathcal{U}}_{\mathfrak{q}} s\ell(2)$ AT AN EVEN ROOT OF UNITY

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ABSTRACT. We show that the full matrix algebra $\mathrm{Mat}_p(\mathbb{C})$ is a $\mathcal{U}$-module algebra for $\mathcal{U} = \overline{\mathcal{U}}_{\mathfrak{q}} s\ell(2)$, a $2p^3$-dimensional quantum $s\ell(2)$ group at the $2p$th root of unity. $\mathrm{Mat}_p(\mathbb{C})$ decomposes into a direct sum of projective $\mathcal{U}$-modules $\mathcal{P}_n^+$ with all odd n, $1 \leqslant n \leqslant p$. In terms of generators and relations, this $\mathcal{U}$-module algebra is described as the algebra of q-differential operators "in one variable" with the relations $\partial z = \mathfrak{q} - \mathfrak{q}^{-1} + \mathfrak{q}^{-2} z \partial$ and $z^p = \partial^p = 0$. These relations define a "parafermionic" statistics that generalizes the fermionic commutation relations. By the Kazhdan–Lusztig duality, it is to be realized in a manifestly quantum-group-symmetric description of $(p,1)$ logarithmic conformal field models. We extend the Kazhdan–Lusztig duality between $\mathcal{U}$ and the $(p,1)$ logarithmic models by constructing a quantum de Rham complex of the new $\mathcal{U}$-module algebra and discussing its field-theory counterpart.

1. INTRODUCTION

1.1. The main results. For an integer $p \geqslant 2$, let $\mathfrak{q} = e^{\frac{i\pi}{p}}$ and let $\mathcal{U} = \overline{\mathcal{U}}_{\mathfrak{q}} s\ell(2)$ be the quantum group with generators E, K, and F and the relations

$$KEK^{-1} = \mathfrak{q}^2 E, \quad KFK^{-1} = \mathfrak{q}^{-2} F, \qquad [E,F] = \frac{K - K^{-1}}{\mathfrak{q} - \mathfrak{q}^{-1}}, \tag{1.1}$$

$$E^p = F^p = 0, \quad K^{2p} = 1 \tag{1.2}$$

(and the Hopf algebra structure to be described below).

We construct a representation of $\mathcal{U}$ on the full matrix algebra $\mathrm{Mat}_p(\mathbb{C})$. For a $p \times p$ matrix $X = (x_{ij})$, $(EX)_{ij}$ is a linear combination of the right and upper neighbors of x_{ij}, and $(FX)_{ij}$ is a linear combination of the left and lower neighbors, with the coefficients shown in the diagrams:

$$E:\quad \begin{tikzcd} \boxed{i-1,j} \arrow[d, "-\frac{\mathfrak{q}^{2(i-j-1)}}{\mathfrak{q}-\mathfrak{q}^{-1}}"'] & \\ \boxed{i,j} & \boxed{i,j+1} \arrow[l, "(\mathfrak{q}-\mathfrak{q}^{-1})^{-1}"'] \end{tikzcd} \qquad\qquad F:\quad \begin{tikzcd} \boxed{i,j-1} \arrow[r, "-\mathfrak{q}^{j-2i}[j-1]"] & \boxed{i,j} \\ & \boxed{i+1,j} \arrow[u, "\mathfrak{q}^{1-i}[i]"'] \end{tikzcd} \tag{1.3}$$

With the necessary modifications at the boundaries, the precise formulas are as follows:

$$E(X) = \frac{1}{\mathfrak{q} - \mathfrak{q}^{-1}} \begin{pmatrix} x_{12} & \dots & x_{i,j+1} & \dots & 0 \\ \vdots & \ddots & \vdots & & \vdots \\ x_{i,2} - \mathfrak{q}^{2(i-2)} x_{i-1,1} & \dots & x_{i,j+1} - \mathfrak{q}^{2(i-j-1)} x_{i-1,j} & \dots & -\mathfrak{q}^{2(i-1)} x_{i-1,p} \\ \vdots & & \vdots & \ddots & \vdots \\ x_{p,2} - \mathfrak{q}^{-4} x_{p-1,1} & \dots & x_{p,j+1} - \mathfrak{q}^{-2(j+1)} x_{p-1,j} & \dots & -\mathfrak{q}^{-2} x_{p-1,p} \end{pmatrix}$$

(with the only zero in the top right corner), *where we explicitly show the ith row and the jth column*;

$$(KX)_{ij} = \mathfrak{q}^{2(i-j)} x_{ij};$$

and

$$F(X) = \begin{pmatrix} x_{21} & \dots & x_{2,j} - \mathfrak{q}^{j-2}[j-1] x_{1,j-1} & \dots & x_{2,p} + \mathfrak{q}^{-2} x_{1,p-1} \\ \vdots & \ddots & \vdots & & \vdots \\ \mathfrak{q}^{1-i}[i] x_{i+1,1} & \dots & \mathfrak{q}^{1-i}[i] x_{i+1,j} - \mathfrak{q}^{j-2i}[j-1] x_{i,j-1} & \dots & \mathfrak{q}^{1-i}[i] x_{i+1,p} + \mathfrak{q}^{-2i} x_{i,p-1} \\ \vdots & & \vdots & \ddots & \vdots \\ 0 & \dots & -\mathfrak{q}^{j}[j-1] x_{p,j-1} & \dots & x_{p,p-1} \end{pmatrix}$$

(with the only zero in the bottom left corner), where we again show the ith row and the jth column, and where

$$[n] = \frac{\mathfrak{q}^n - \mathfrak{q}^{-n}}{\mathfrak{q} - \mathfrak{q}^{-1}}.$$

Theorem.

(1) *The above formulas define a representation of* $\mathcal{U} = \overline{\mathcal{U}}_{\mathfrak{q}} s\ell(2)$ *on* $\mathrm{Mat}_p(\mathbb{C})$.
(2) $\mathrm{Mat}_p(\mathbb{C})$ *is a* $\mathcal{U}$*-module algebra.*

We recall that for a Hopf algebra $\mathcal{H}$, an $\mathcal{H}$-module algebra is an algebra in the tensor category of $\mathcal{H}$-modules, i.e., is a (left) $\mathcal{H}$-module V with a composition law $V \otimes V \to V$ such that $h(vw) = \sum h'(v) h''(w)$ for $h \in \mathcal{H}$ and $v, w \in V$ (here, $\Delta(h) = \sum h' \otimes h''$ is Sweedler's notation for coproduct).[1]

The quantum group $\mathcal{U}$ has $2p$ irreducible representations $\mathcal{X}_r^{\pm}$, $1 \leqslant r \leqslant p$, with $\dim \mathcal{X}_r^{\pm} = r$ [1]. We let $\mathcal{P}_r^{\pm}$ denote their projective covers. The "plus" representations are distinguished from the "minus" ones by the fact that tensor products $\mathcal{X}_r^+ \otimes \mathcal{X}_s^+$ decompose into the $\mathcal{X}_{r'}^+$ and $\mathcal{P}_{r'}^+$ (and $\mathcal{X}_1^+$ is the trivial representation).

Theorem (continued)**.**

(3) $\mathrm{Mat}_p(\mathbb{C})$ *decomposes into indecomposable* $\mathcal{U}$*-modules as*

$$\mathrm{Mat}_p(\mathbb{C}) = \mathcal{P}_1^+ \oplus \mathcal{P}_3^+ \oplus \dots \oplus \mathcal{P}_\nu^+, \tag{1.4}$$

[1]In simple words, the condition states a natural compatibility between the $\mathcal{H}$-action and multiplication on V, "natural" because $\mathcal{H}$ acts on a product via comultiplication. Claim (2) is thus that the standard matrix multiplication is compatible with the proposed action of $\mathcal{U}$ (and its comultiplication).

where $\nu = p$ is p if odd and $p-1$ if p is even.

The algebra in (1.4) is the smallest $\mathcal{U}$-module algebra that contains $\mathcal{P}_1^+$, the projective cover of the trivial representation. This $2p$-dimensional module can be visualized as a span of $2p$ elements with the $\mathcal{U}$-action given by [1]

$$\begin{tikzcd}
& t \arrow[dl, "E"'] \arrow[dr, "F"] & \\
\ell_{p-1} \rightleftarrows \ell_{p-2} \rightleftarrows \ldots \rightleftarrows \ell_1 \arrow[dr, "F"'] & & r_1 \rightleftarrows \ldots \rightleftarrows r_{p-2} \rightleftarrows r_{p-1} \arrow[dl, "E"] \\
& 1 &
\end{tikzcd}$$

where the horizontal arrows represent the action of E (to the left) and F (to the right) up to *nonzero* factors and the tilted arrows indicate that the map in the opposite direction vanishes; the bottom 1 spans the 1-dimensional submodule. In the *algebra* defined on the sum of projective modules, we can say more about the structure of $\mathcal{P}_1^+$.

Theorem (continued)**.**

(4) *There is an isomorphism of $\mathcal{U}$-module algebras*

$$\mathcal{P}_1^+ \oplus \mathcal{P}_3^+ \oplus \cdots \oplus \mathcal{P}_\nu^+ \cong \overline{\mathbb{C}}_{\mathfrak{q}}[z, \partial],$$

where $\overline{\mathbb{C}}_{\mathfrak{q}}[z, \partial]$ is the associative algebra with generators ∂ and z and the relations

$$\partial z = \mathfrak{q} - \mathfrak{q}^{-1} + \mathfrak{q}^{-2} z \partial, \tag{1.5}$$

$$\partial^p = 0, \qquad z^p = 0. \tag{1.6}$$

(5) *Under this isomorphism, the "wings" of the projective module $\mathcal{P}_1^+$ in* (1.4) *are powers of a single generator each,*

$$\begin{tikzcd}
& t \arrow[dl, "E"'] \arrow[dr, "F"] & \\
z^{p-1} \rightleftarrows z^{p-2} \rightleftarrows \ldots \rightleftarrows z \arrow[dr, "F"'] & & \partial \rightleftarrows \ldots \rightleftarrows \partial^{p-2} \rightleftarrows \partial^{p-1} \arrow[dl, "E"] \\
& 1 &
\end{tikzcd} \tag{1.7}$$

and the "top" element is

$$t = \sum_{i=1}^{p-1} \frac{1}{[i]} z^i \partial^i. \tag{1.8}$$

In other words, our $\mathcal{U}$-module algebra is identified with the algebra of q-differential operators "in one variable" with nilpotency conditions (1.6) (and with a slightly unusual rule for carrying ∂ through z). This is parallel to the classic result that $\mathrm{Mat}_p(\mathbb{C})$ is generated by x and y satisfying the relations $yx = qxy$ and $x^p = y^p = 1$, where q is the pth root of unity [2] (a finite quantum plane in the modern terminology), but there seems to be no direct ("exponential") relation between our "nilpotent" ($\partial^p = z^p = 0$) and the classic

"unipotent" ($x^p = y^p = 1$) constructions. Apart from matrix curiosities, the q-differential operators yield a preferential ("more invariant") description of the algebra on the sum of "odd" projective $\mathcal{U}$-modules $\mathcal{P}_1^+ \oplus \mathcal{P}_3^+ \oplus \dots$ compared with its matrix realization.

Obviously, t in (1.7) is defined up to the addition of $\alpha 1$, $\alpha \in \mathbb{C}$, and expression (1.8) is therefore a particular representative of this class; this is to be understood in what follows.

For a quasitriangular $\mathcal{H}$, an $\mathcal{H}$-module algebra is said to be quantum commutative [3] (also, $\mathcal{H}$-, R-, or braided commutative) if

$$vw = \sum R^{(2)}(w)\, R^{(1)}(v), \tag{1.9}$$

for all $v, w \in V$, where $R = \sum R^{(1)} \otimes R^{(2)} \in \mathcal{H} \otimes \mathcal{H}$ is the universal R-matrix. Our $\mathcal{U}$-module algebra is not quantum commutative; nevertheless, *relation* (1.9) *is satisfied for* $v = z^i \partial^j$ *and* $w = z^m \partial^n$ *if and only if either* $n = 0$ *or* $i = 0$ *or* $|i + m - j - n| \geqslant p$.

Returning to matrices and representing commutation relations (1.5) as[2]

$$z = \begin{pmatrix} 0 & \dots\dots\dots\dots & & & 0 \\ 1 & 0 & \dots\dots & & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \dots\dots & & 1 & 0 \end{pmatrix}, \qquad \partial = (\mathfrak{q} - \mathfrak{q}^{-1}) \begin{pmatrix} 0 & 1 & \dots\dots\dots & & 0 \\ 0 & 0 & \mathfrak{q}^{-1}[2] & \dots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \dots\dots\dots & & 0 & \mathfrak{q}^{2-p}[p-1] \\ 0 & \dots\dots\dots & & 0 & 0 \end{pmatrix}, \tag{1.10}$$

we have one of the "matrix curiosities"—*integers rather than q-integers* in the matrix representation of (1.8):

$$t = (\mathfrak{q} - \mathfrak{q}^{-1}) \begin{pmatrix} 0 & 0 & \dots\dots\dots\dots & & & 0 \\ 0 & 1 & 0 & \dots\dots\dots & & 0 \\ 0 & 0 & 2 & 0 & \dots\dots & 0 \\ \vdots & & & \ddots & & \vdots \\ 0 & \dots & & 0 & p-2 & 0 \\ 0 & \dots\dots\dots & & & 0 & p-1 \end{pmatrix}. \tag{1.11}$$

Next, it turns out that a differential calculus can be developed for our algebra $\overline{\mathbb{C}}_{\mathfrak{q}}[z, \partial]$ such that the differential (satisfying the "classical" Leibnitz rule!) commutes with the quantum group action. Let $\mathbb{C}_{\mathfrak{q}}[\zeta, \delta]$ be an "odd" counterpart of $\overline{\mathbb{C}}_{\mathfrak{q}}[z, \partial]$—the algebra on ζ and δ with the relations $\zeta^2 = 0$, $\delta^2 = 0$, and $\delta\,\zeta = -\mathfrak{q}^{-2}\,\zeta\,\delta$. The new variables are to be considered the differentials of the "coordinates," $\zeta = d(z)$ and $\delta = d(\partial)$.[3]

Theorem (continued)**.**

(6) *A quotient of* $\overline{\mathbb{C}}_{\mathfrak{q}}[z, \partial] \otimes \mathbb{C}_{\mathfrak{q}}[\zeta, \delta]$ *can be endowed with the structure of a dif-*

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[2]We do not reduce the expressions using that $\mathfrak{q}^p = -1$ and $[p-i] = [i]$ when the unreduced form helps to see a pattern.

[3]If our $\mathbb{C}_{\mathfrak{q}}[z, \partial]$ is relabeled as $\mathbb{C}_{\mathfrak{q}}^{2|0}[z, \partial]$, then its "odd" counterpart is to be denoted as $\mathbb{C}_{\mathfrak{q}}^{0|2}[\zeta, \delta]$; we use the simpler notation for brevity.

ferential $\mathcal{U}$-module algebra $(\Omega\overline{\mathbb{C}}_{\mathfrak{q}}[z,\partial],d)$ that is a quantum de Rham complex of $\overline{\mathbb{C}}_{\mathfrak{q}}[z,\partial]$.

The notation $\Omega\overline{\mathbb{C}}_{\mathfrak{q}}[z,\partial]$ assumes that $\overline{\mathbb{C}}_{\mathfrak{q}}[z,\partial]$ is the algebra of 0-forms. The exact formulas defining the quotient and the $\mathcal{U}$-action are given in Sec. 4 below.

As an illustration of the action of the differential d on the module structure, we note that the unity in $\mathcal{P}_1^+$, Eq. (1.7), is annihilated, and therefore $\mathcal{P}_1^+$ is not preserved by d. On the other hand, there is another, not d-closed element in $\Omega^1\overline{\mathbb{C}}_{\mathfrak{q}}[z,\partial]$ in the same grade as dt, *and* elements in the cohomology of d, which together with $d(\mathcal{P}_1^+)$ arrange into the direct sum of two $\mathcal{U}$-modules

$$\begin{tikzcd}[column sep=small]
z^{p-1}\zeta \arrow[dr, "F"] & & & & & & & \sum\limits_{i=1}^{p-1}\frac{1}{[i]}d(z^i)\,\partial^i \arrow[dl, "E"'] \\
& z^{p-2}\zeta & \rightleftarrows\dots\rightleftarrows & z\zeta & \rightleftarrows & \zeta & & \\
& & & & & \bigoplus & & \\
& & & & \sum\limits_{i=1}^{p-1}\frac{1}{[i]}z^i\,d(\partial^i) \arrow[dr, "F"] & & & \partial^{p-1}\delta \arrow[dl, "E"'] \\
& & & & & \delta\ \rightleftarrows\ \partial\delta\ \rightleftarrows\dots\rightleftarrows & \partial^{p-2}\delta &
\end{tikzcd} \tag{1.12}$$

where $d(z^i)=\mathfrak{q}^{1-i}[i]\,z^{i-1}\,\zeta$ and $d(\partial^i)=\mathfrak{q}^{i-1}[i]\,\partial^{i-1}\,\delta$. *The "corners" $z^{p-1}\,\zeta$ and $\partial^{p-1}\,\delta$ are in the cohomology of the differential.*

1.2. Motivation and some (un)related approaches. Our interest in the quantum group $\mathcal{U}=\overline{\mathcal{U}}_{\mathfrak{q}}s\ell(2)$ and related objects stems from its occurrence in logarithmic conformal field theories [1, 4] (also see a similar quantum group structure in [5, 6], a review in [7], and a further development in [8]).[4] But this particular version of the quantum $s\ell(2)$ actually made its first appearance much earlier; a regrettable omission in (the arXiv version of) [7] was paper [21], where the regular representation of $\mathcal{U}$ was elegantly described in terms of the even subalgebra of a matrix algebra times a Grassmann algebra on two generators for each block (also see [22, 23, 24] for a very closely related quantum group at $p=3$; our quantum group was also the subject of attention in [25, 26]).

The correspondence between $\mathcal{U}$ and the $(p,1)$ logarithmic conformal field models, which is a version of the Kazhdan–Lusztig duality [27], extends not only to the repre-

[4]On the subject of logarithmic $(p,1)$ models, without attempting to be complete in any way, we note the pioneering works [9, 10, 11] (where, in particular, the symmetry of the model — the *triplet algebra* — was identified), reviews [12, 13] of the early stages, "logarithmic deformations" in [14], the definition of the triplet algebra $W(p)$ at general p as the kernel of a screening and the fusion algebra of the $2p$ irreducible $W(p)$-representations [15] (also see [16]), the study of $W(p)$ with the aid of Zhu's algebra [17], interesting recent advances in [18, 19, 20, 8], and, of course, the numerous references therein.

sentation theories but also to modular group actions, the modular group action generated from the characters of the $W(p)$ algebra being isomorphic to that on the quantum group center [1, 4, 5, 6]. But algebraic structures on $\mathcal{U}$-modules have not been investigated in the Kazhdan–Lusztig context. Relations (1.5), (1.6) are in fact a quantum-group counterpart of the "hidden" quantum-group symmetry of the $(p,1)$ logarithmic conformal model (see **1.3** below).

On the other hand, commutation relation (1.5) can be compared with the (considerably more general) setting of quantum Weyl algebras [28, 29, 30]. There, one considers the defining relations (the ∂^j are not powers of an element but different elements)

$$\sum R_{ij}^{kl} x_k x_l = q x_i x_j,$$
$$\partial^j x_i = \delta_i^j + q \sum R_{il}^{jk} x_k \partial^l, \quad 1 \leqslant i, j, \ldots \leqslant n,$$
$$\sum R_{kl}^{ij} \partial^k \partial^l = q \partial^i \partial^j,$$

where R is an $n^2 \times n^2$ matrix solution of the Yang–Baxter equation *and* the Hecke relation. For the "$g\ell_n$" R-matrix, in particular,

$$\partial^i x_i = 1 + q^2 x_i \partial^i + (q^2 - 1) \sum_{j>i} x_j \partial^j,$$

which in the case $n = 1$ (of little interest in the general theory of quantum Weyl algebras) becomes

$$\partial x = 1 + q^2 x \partial.$$

Our relation (1.5) involves $\mathfrak{q} - \mathfrak{q}^{-1}$ instead of unity, which is dictated by the $\mathcal{U}$-module algebra property, with $\mathcal{U} = \overline{\mathcal{U}}_{\mathfrak{q}} s\ell(2)$ being our main, initial object (in contrast to quantum Weyl algebras, where the "$\partial\,x$–$x\,\partial$" relations are considered primary and then quantum enveloping algebras generated by the $x_i \partial^j$ are studied; also, our R-matrix does not satisfy the Hecke relation).

1.3. "Parafermionic statistics".

1.3.1. Relations (1.5) and (1.6) take a "fermionic" form for $p = 2$:

$$\{\partial, \partial\} = 0, \quad \{z, z\} = 0, \quad \{\partial\,, z\} = 2i,$$

where $\{\ ,\ \}$ is the anticommutator.[5] This "fermionic statistics" (i.e., Clifford-algebra relations) is very well known to be relevant to the simplest logarithmic conformal field theory model in the $(p,1)$ family, the $(2,1)$ model [10, 11], whose dual quantum group is our $\mathcal{U}$ at $p = 2$ ($\mathfrak{q} = i$): this model is described by "symplectic fermions" — conformal fields defined on the complex plane that satisfy the fermionic commutation relations [31]. For

[5] These three anticommutators are not unrelated to, but must be clearly distinguished from the relations in the $\mathcal{U}$ algebra itself at $p = 2$, which can be written as $\{E, E\} = 0$, $\{\tilde{F}, \tilde{F}\} = 0$, and $\{E, \tilde{F}\} = \frac{1}{2i}(1 - K^2)$ for $\tilde{F} = KF$.

general p, the $(p,1)$ logarithmic model corresponds under the Kazhdan–Lusztig duality just to $\mathcal{U}$ at $\mathfrak{q} = e^{\frac{i\pi}{p}}$, and relations (1.5) and (1.6) are a generalization of the fermionic statistics.

1.3.2. Manifestly quantum-group-invariant description of LCFTs. For $p > 2$, an important problem is to describe the $(p,1)$ logarithmic conformal models in *manifestly quantum-group-invariant terms*. The idea of an explicit quantum group symmetry was (somewhat implicitly) expressed in [4], where the Fermi statistics realized for $p = 2$ was predicted to extend for general p to a "parafermionic"[6] statistics on $p-1$ pairs of fields, which would also allow realizing projective modules over the triplet algebra.

Relations (1.5) and (1.6) suggest this general-p, "parafermionic" statistics of the $(p,1)$ logarithmic conformal field theory models. To realize it, we introduce $p-1$ pairs of fields $\zeta^m(w)$ and $\delta^m(w)$, $m = 1,\dots,p-1$, carrying the same $\mathcal{U}$ representation as the z^m and ∂^m, and set $\delta^0(w) = \zeta^0(w) = 1$ (here, w is a coordinate on the complex plane). The $\zeta^m(w)$ and $\delta^m(w)$ have conformal weight zero. With (1.7) rewritten in terms of the fields,

(1.13)
$$\begin{tikzcd}[column sep=small]
 & \Lambda(w) = \sum\limits_{n=1}^{p-1} \frac{1}{[n]}\zeta^n\delta^n(w) \arrow[dl, "E"'] \arrow[dr, "F"] & \\
\zeta^{p-1}(w) \rightleftarrows \ldots \rightleftarrows \zeta^1(w) \arrow[dr, "F"'] & & \delta^1(w) \rightleftarrows \ldots \rightleftarrows \delta^{p-1}(w), \arrow[dl, "E"] \\
 & 1 &
\end{tikzcd}$$

it follows that $\Lambda(w)$ is a *logarithmic partner* of the identity operator (cf. [4]).

1.3.3. First-order "parafermionic" systems. The differential d acting on conformal fields (in terms of the coordinate w on the complex plane) commutes with the quantum group action on the fields. This is also the case with d in the de Rham complex $\Omega\overline{\mathbb{C}}_{\mathfrak{q}}[z,\partial]$ on the algebraic side, and we do not therefore distinguish the two differentials. It is instructive to rewrite (1.12) in terms of fields. For this, we introduce the fields $\eta^n(w)$ as

(1.14)
$$d\delta^n(w) = [n]\mathfrak{q}^{n-1}\eta^n(w), \quad n = 1,\dots,p-1.$$

Then the fields $\zeta^n(w)$ and $\eta^n(w)$ constitute a $(p-1)$-component "parafermionic" version of the first-order fermionic system. The field realization of one of the modules in (1.12) is

(1.15)
$$\begin{tikzcd}[column sep=small]
\mathcal{J}(w) \equiv \sum\limits_{n=1}^{p-1} \mathfrak{q}^{n-1}\zeta^n\eta^n(w) \arrow[dr, "F"] & & e^{\sqrt{2p}\,\varphi(w)} \arrow[dl, "E"'] \\
 & \eta^1(w) \rightleftarrows \ldots \rightleftarrows \eta^{p-1}(w), &
\end{tikzcd}$$

[6] The word "parafermionic" is somewhat overloaded here (and, in particular, is not related to the parafermions discussed in the context of logarithmic conformal field theories in [32]); although its use is motivated by the discussion in [33], "anyonic" might be a better choice.

where $\varphi(w)$ is introduced as $d\varphi(w) = \mathcal{I}(w)$,

$$\mathcal{I}(w) = \sum_{n=1}^{p-1} \tfrac{1}{[n]} d\zeta^n \delta^n(w), \tag{1.16}$$

and $e^{\sqrt{2p}\,\varphi(w)}$ is the "screening current"—a field on the complex plane such that taking the first-order pole in the OPE with it defines a screening operator.

In the Appendix, we consider the "parafermionic" fields, generalizing free fermions, in more detail. The extension from fermions ($p = 2$) to "parafermions" (general p) is also closely related to an algebraic pattern that we now recall.

1.3.4. On the algebraic side, just the same ideology of a "quantum" generalization of fermionic commutation relations was put forward in [3]. The guiding principle was that of quantum commutativity, which "encompasses commutativity of algebras and superalgebras on one hand and the quantum planes and superplanes on the other" [3]. A number of examples, including the quantum plane, were considered in that paper. We also note the related points in [34, 35]; in particular, a free algebra on the ξ_i with the relations

$$\xi_i \xi_j = R_{ij}^{mn} \xi_m \xi_n$$

(where R_{ij}^{mn} is again a matrix solution of the Yang–Baxter equation) is quantum commutative in the category of Yetter–Drinfeld modules over the bialgebra obtained from R via the Faddeev–Reshetikhin–Takhtajan construction, i.e., the free algebra on the c^i_j with the relations

$$R^{ij}_{mn} c^n_k c^m_l = R^{mn}_{lk} c^i_m c^j_n.$$

(A partly reversed logic has also been used to find solutions of the Yang–Baxter equation from Yetter–Drinfeld ("Yang–Baxter") modules [36]).

For us, as in [1, 7], the quantum group $\mathcal{U}$ is not reconstructed from some R-matrix but is given as the primary object (originally determined by the Kazhdan–Lusztig duality with logarithmic conformal field theory). We then define an algebra on ∂ and z with the crucial commutation relation given by (1.5), verify the $\mathcal{U}$-module property, and find the algebra decomposition. Alternatively, it could be possible to start with the appropriate sum of (the "odd") projective quantum-group modules and conclude somehow that it is an associative algebra; from this perspective, the results in this paper include finding the generators (∂ and z) and relations ((1.5) and (1.6)) in this associative algebra.

1.4. $\overline{\mathcal{U}}_{\mathfrak{q}} s\ell(2)$. We quote several results about our quantum group $\mathcal{U}$ in (1.1), (1.2) [1].

The Hopf algebra structure of $\mathcal{U}$ is given by

$$\Delta(E) = E \otimes K + 1 \otimes E, \quad \Delta(K) = K \otimes K, \quad \Delta(F) = F \otimes 1 + K^{-1} \otimes F,$$
$$\varepsilon(E) = \varepsilon(F) = 0, \quad \varepsilon(K) = 1,$$
$$S(E) = -EK^{-1}, \quad S(K) = K^{-1}, \quad S(F) = -KF.$$

Therefore, in particular, the condition for an algebra V carrying a representation of $\mathcal{U}$ to be a $\mathcal{U}$-module algebra is that

$$\begin{aligned}E(vw) &= (Ev)(Kw) + v(Ew),\\ K(vw) &= (Kv)(Kw),\\ F(vw) &= F(v)w + (K^{-1}v)Fw\end{aligned}$$

for $v, w \in V$.

For each $1 \leqslant r \leqslant p-1$, the projective module $\mathcal{P}_r^{\pm}$ that covers the irreducible representation $\mathcal{X}_r^{\pm}$ has dimension $2p$; for $r = p$, the projective module coincides with the irreducible representation [1]. The structure of projective $\mathcal{U}$-modules is made very explicit in [1] and all the indecomposable representations of $\mathcal{U}$ are classified in [4] (they can also be deduced from a more general approach in [37]).

The universal R-matrix for $\mathcal{U}$ was found in [1]:

$$R = \frac{1}{4p}\sum_{i=0}^{p-1}\sum_{a,b=0}^{4p-1}\frac{(\mathfrak{q}-\mathfrak{q}^{-1})^i}{[i]!}\mathfrak{q}^{\frac{i(i-1)}{2}+i(a-b)-\frac{ab}{2}}E^iK^{\frac{a}{2}}\otimes F^iK^{\frac{b}{2}}. \tag{1.17}$$

Strictly speaking, this is not an R-matrix *for the quantum group* $\mathcal{U}$ because of the half-integer powers of K involved here. This was discussed in detail in [1]; an essential point is that the so-called monodromy matrix $M = R_{21}R$ *is* an element of $\mathcal{U}\otimes\mathcal{U}$; in our present context, a similar effect is that we do not have to introduce half-integer powers of $\mathfrak{q}$ because all eigenvalues of K, which are $\mathfrak{q}^n$, occur with even n here. Thus, whenever K acts by $\mathfrak{q}^{2n} = e^{\frac{2i\pi n}{p}}$, $0 \leqslant n \leqslant p-1$, we set $K^{\frac{1}{2}}$ to act by $\mathfrak{q}^n = e^{\frac{i\pi n}{p}}$.

The $\mathfrak{q}$-integers $[n]$ were defined above, and we also use the standard notation

$$[n]! = [1][2]\dots[n], \qquad \begin{bmatrix}m\\n\end{bmatrix} = \frac{[m]!}{[m-n]!\,[n]!}$$

(with $\begin{bmatrix}m\\n\end{bmatrix} = 0$ for $m < n$).

Most of the material that relates to proving the theorem is collected in Sec. 2; some remarks about the matrix realization are in Sec. 3; the extension to a differential algebra (the quantum de Rham complex of $\overline{\mathbb{C}}_{\mathfrak{q}}[z,\partial]$) is given in Sec. 4. Implications of the "parafermionic statistics" (i.e., of the commutation relations in our $\mathcal{U}$-module algebra) for conformal field theory are discussed in the Appendix.

2. q-DIFFERENTIAL OPERATORS ON THE LINE AT A ROOT OF UNITY

We consider the "quantum line" $\mathbb{C}[z]$, i.e., the space of polynomials in one variable; the "quantum" (i.e., noncommutative) features are to be seen not in the polynomials themselves but in operators acting on them (and therefore a *quantum* line is a certain abuse of speech unless it is endowed with some extra structures).

2.1. z, ∂, and a $\mathcal{U}$ action.

2.1.1. We define the $\mathcal{U}$ action on $\mathbb{C}[z]$ as

$$\begin{aligned}
E\,z^m &= -\mathfrak{q}^m[m]z^{m+1},\\
K\,z^m &= \mathfrak{q}^{2m}\,z^m,\\
F\,z^m &= [m]\mathfrak{q}^{1-m}\,z^{m-1}.
\end{aligned}$$

That this is indeed a $\mathcal{U}$ action is easy to verify. Clearly, the unity spans a submodule. The module structure of $\mathbb{C}[z]$ is given by the diagram (an infinite version of the zigzag modules considered in [4]; see also [37])

$$\begin{tikzcd}[column sep=small]
 & \cdots & z^{2p+1} \arrow[dr, "F"'] & & z^{2p-1} \arrow[dl, "E"] \arrow[r, shift left] & \cdots \arrow[l, shift left] \arrow[r, shift left] & z^{p+1} \arrow[l, shift left] \arrow[dr, "F"'] & & z^{p-1} \arrow[dl, "E"] \arrow[r, shift left] & \cdots \arrow[l, shift left] \arrow[r, shift left] & z \arrow[l, shift left] \arrow[d, "F"] \\
\cdots & & & z^{2p} & & & & z^{p} & & & 1
\end{tikzcd}$$

where the horizontal $\rightleftarrows$ arrows denote the action by F (to the right) and E (to the left) up to nonzero factors.

2.1.2. The formulas above actually make $\mathbb{C}[z]$ into a $\mathcal{U}$-module algebra. The elementary proof of this fact amounts to the calculation

$$\begin{aligned}
\sum E'(z^m)\,E''(z^n) = z^m E(z^n) + E(z^m)K(z^n) &= -\mathfrak{q}^n[n]z^m\,z^{n+1} - \mathfrak{q}^m[m]z^{m+1}\mathfrak{q}^{2n}z^n\\
&= -(\mathfrak{q}^n[n] + \mathfrak{q}^{m+2n}[m])z^{m+n+1} = -\mathfrak{q}^{m+n}[m+n]z^{m+n+1} = E(z^{m+n}),
\end{aligned}$$

and similarly for F.

2.1.3. We next introduce a "dual" quantum line $\mathbb{C}[\partial]$ of polynomials in a q-derivative operator ∂ on $\mathbb{C}[z]$, and postulate the commutation relation (1.5). A simple exercise in recursion then leads to the relations

$$\partial^m\,z^n = \sum_{i\geqslant 0}\mathfrak{q}^{-(2m-i)n+im-\frac{i(i-1)}{2}}\begin{bmatrix}m\\ i\end{bmatrix}\begin{bmatrix}n\\ i\end{bmatrix}[i]!\,\left(\mathfrak{q}-\mathfrak{q}^{-1}\right)^i z^{n-i}\partial^{m-i}$$

(because of the $\mathfrak{q}$-binomial coefficients, the range of i is bounded above by $\min(m,n)$). Anticipating the result in (1.7), we thus have the commutation relations between elements of the projective module $\mathcal{P}^+_1$.

We let $\mathbb{C}_{\mathfrak{q}}[z,\partial]$ denote the associative algebra generated by z and ∂ with relation (1.5). In the formulas such as above, z is the operator of multiplication by z, and all expressions like $\partial^m z^n$ are understood accordingly; as regards the *action* of ∂ on $\mathbb{C}[z]$, it is given by the $i = m$ term in the last formula:

$$\partial^m(z^n) = \mathfrak{q}^{m(m-n)+\frac{m(m-1)}{2}}\begin{bmatrix}n\\ m\end{bmatrix}[m]!\,\left(\mathfrak{q}-\mathfrak{q}^{-1}\right)^m z^{n-m}.$$

2.1.4. It follows from **2.1.3** that

$$\partial^m z = \mathfrak{q}^{-2m} z \partial^m + \mathfrak{q}(1-\mathfrak{q}^{-2m})\partial^{m-1}$$

and

$$\partial z^n = \mathfrak{q}^{-2n} z^n \partial + \mathfrak{q}(1-\mathfrak{q}^{-2n}) z^{n-1},$$

and hence ∂^p *and* z^p *are central in* $\mathbb{C}_{\mathfrak{q}}[z,\partial]$.

We note that Lusztig's trick of resolving the ambiguities in $X \mapsto (\partial^p X - X\partial^p)/[p]$ and $X \mapsto (z^p X - X z^p)/[p]$ then yields two *derivations* of $\mathbb{C}_{\mathfrak{q}}[z,\partial]$:

$$\mathfrak{d}: \begin{array}{l} z^n \mapsto \displaystyle\sum_{i=1}^{n} (-1)^i \mathfrak{q}^{in-\frac{i(i-1)}{2}} \frac{[n-i+1]\dots[n]}{[i]} \left(\mathfrak{q}-\mathfrak{q}^{-1}\right)^i z^{n-i}\partial^{p-i}, \\ \partial^n \mapsto 0 \end{array}$$

and

$$\mathfrak{z}: \begin{array}{l} z^n \mapsto 0, \\ \partial^n \mapsto -\displaystyle\sum_{i=1}^{n} (-1)^i \mathfrak{q}^{in-\frac{i(i-1)}{2}} \frac{[n-i+1]\dots[n]}{[i]} \left(\mathfrak{q}-\mathfrak{q}^{-1}\right)^i z^{p-i}\partial^{n-i}. \end{array}$$

2.1.5. We next define the $\mathcal{U}$ action on $\mathbb{C}[\partial]$ as

$$\begin{aligned} E\,\partial^n &= \mathfrak{q}^{1-n}[n]\partial^{n-1}, \\ K\,\partial^n &= \mathfrak{q}^{-2n}\partial^n, \\ F\,\partial^n &= -\mathfrak{q}^n[n]\partial^{n+1}. \end{aligned}$$

Clearly, this *is* a $\mathcal{U}$ action, the unity $1 = \partial^0$ is a submodule, and this action makes $\mathbb{C}[\partial]$ into a $\mathcal{U}$-module algebra.

2.1.6. Lemma. $\mathbb{C}_{\mathfrak{q}}[z,\partial]$ *is a* $\mathcal{U}$*-module algebra.*

The proof amounts to verifying that E and F preserve the ideal generated by the left-hand side of (1.5):

$$\begin{aligned} E(\partial z - (\mathfrak{q}-\mathfrak{q}^{-1}) - \mathfrak{q}^{-2} z\partial) &= E(\partial)Kz + \partial E(z) - \mathfrak{q}^{-2}(E(z)K(\partial) + zE(\partial)) \\ &= \mathfrak{q}^2 z - \mathfrak{q}\partial z^2 - \mathfrak{q}^{-2}(-\mathfrak{q}z^2\mathfrak{q}^{-2}\partial + z) = 0 \end{aligned}$$

by **2.1.3**. Similarly,

$$\begin{aligned} F(\partial z - (\mathfrak{q}-\mathfrak{q}^{-1}) - \mathfrak{q}^{-2} z\partial) &= K^{-1}(\partial)F(z) + F(\partial)z - \mathfrak{q}^{-2}(K^{-1}(z)F\partial + F(z)\partial) \\ &= \mathfrak{q}^2\partial - \mathfrak{q}\partial^2 z - \mathfrak{q}^{-2}(-\mathfrak{q}^{-2}z\mathfrak{q}\partial^2 + \partial) = 0 \end{aligned}$$

by **2.1.3** as well.

2.1.7. As noted in the Introduction, the quantum commutativity property, Eq. (1.9), is violated for our $\mathcal{U}$-module algebra; for example, we have

$$\sum R^{(2)}(\partial)\, R^{(1)}(z) = \sum_{j=0}^{p-1} \gamma_j\, z^j \partial^j$$

with the nonzero coefficients

$$\gamma_j = \sum_{i=\max(j-1,0)}^{j+p-2} (\mathfrak{q}-\mathfrak{q}^{-1})^{2i-j+1} \mathfrak{q}^{-i^2-4i-2-\frac{1}{2}(j^2+3j)-ij} \begin{bmatrix} i+1 \\ j \end{bmatrix}^2 [i]!\,[i+1-j]!\,.$$

Yet in the basis of monomials $z^m\partial^n$, Eq. (1.9) holds in the cases noted above, which in particular include all pairs $v = z^i\partial^j$, $w = z^m$ and all pairs $v = \partial^j$, $w = z^m\partial^n$, for which all the ∂^n in wv stand to right of the z^m. For example, with the R-matrix in (1.17), we calculate

$$R(\partial \otimes z) = \sum_{i=0}^{1} \frac{(\mathfrak{q}-\mathfrak{q}^{-1})^i}{[i]!}\, \mathfrak{q}^{\frac{i(i-1)}{2}-2(i-1)^2} E^i\partial \otimes F^i z = \mathfrak{q}^{-2}\partial \otimes z + (\mathfrak{q}-\mathfrak{q}^{-1}) 1 \otimes 1,$$

and therefore the right-hand side of (1.9) evaluates as

$$\sum R^{(2)}(z)\, R^{(1)}(\partial) = \mathfrak{q}-\mathfrak{q}^{-1} + \mathfrak{q}^{-2} z\partial = \partial z.$$

In the commutative subalgebras $\mathbb{C}[z]$ and $\mathbb{C}[\partial]$, even simpler,

$$R(z \otimes z) = \sum_{i=0}^{1} \frac{(\mathfrak{q}-\mathfrak{q}^{-1})^i}{[i]!}\, \mathfrak{q}^{\frac{i(i-1)}{2}-2(i^2-1)} E^i z \otimes F^i z = \mathfrak{q}^2 z \otimes z + (\mathfrak{q}-\mathfrak{q}^{-1})(-\mathfrak{q}) z^2 \otimes 1,$$

which makes (1.9) an identity, and similarly for $R(\partial \otimes \partial)$.

2.2. The quotient $\overline{\mathbb{C}}_{\mathfrak{q}}[z,\partial]$. We saw in **2.1.4** that z^p and ∂^p are central in $\mathbb{C}_{\mathfrak{q}}[z,\partial]$. The formulas for the $\mathcal{U}$ action also imply that $Ez^p = Fz^p = E\partial^p = F\partial^p = 0$. We can therefore take the quotient of $\mathbb{C}_{\mathfrak{q}}[z,\partial]$ by relations (1.6). The quotient $\mathcal{U}$-module algebra is denoted by $\overline{\mathbb{C}}_{\mathfrak{q}}[z,\partial]$ in what follows.

We note that the derivations in **2.1.4** do not descent to $\overline{\mathbb{C}}_{\mathfrak{q}}[z,\partial]$ because, for example, $\mathfrak{d}(z^p) = p(\mathfrak{q}-\mathfrak{q}^{-1})\, 1$.

2.3. The $\mathcal{U}$ action on $\mathbb{C}[z]/z^p$ in terms of q-differential operators. This subsection is a digression not needed in the rest of this paper.

2.3.1. "Scaling" operator $\mathcal{E}$. The operator

$$\mathcal{E} = \frac{\partial z - z\partial}{\mathfrak{q}-\mathfrak{q}^{-1}} = 1 - \mathfrak{q}^{-1} z\partial,$$

commutes with z and ∂ as

$$\mathcal{E} z^n = \mathfrak{q}^{-2n} z^n \mathcal{E}, \quad \mathcal{E}\partial^n = \mathfrak{q}^{2n}\partial^n \mathcal{E}.$$

In what follows, when we speak of the *action* of q-differential operators on $\mathbb{C}[z]$, it is of course understood that $\mathcal{E}(z^n) = \mathfrak{q}^{-2n} z^n$.

We also calculate

$$\mathcal{E}^n = 1 + \sum_{i=1}^{n} \begin{bmatrix} n \\ i \end{bmatrix} (-1)^i \mathfrak{q}^{-ni} z^i \partial^i.$$

In particular, $\mathcal{E}^p = 1 + z^p \partial^p$, and hence

$$\mathcal{E}^p = 1 \quad \text{in} \quad \overline{\mathbb{C}}_{\mathfrak{q}}[z,\partial].$$

Therefore, $\mathcal{E}$ is invertible in $\overline{\mathbb{C}}_{\mathfrak{q}}[z,\partial]$. Moreover, it is easy to see that in $\overline{\mathbb{C}}_{\mathfrak{q}}[z,\partial]$, the above formula for $\mathcal{E}^n$ extends to negative n as

$$\mathcal{E}^n = 1 + \sum_{i=1}^{p-1} \frac{[n-i+1]\dots[n]}{[i]!} (-1)^i \mathfrak{q}^{-ni} z^i \partial^i, \quad n \in \mathbb{Z},$$

which thus gives an explicit representation for $\mathcal{E}^{-1}$, in particular.

The next lemma shows that, as could be expected, the E and F generators acting on $\mathbb{C}[z]/z^p$ are (almost) given by multiplication by z and by a q-derivative.

2.3.2. Lemma. *The $\mathcal{U}$ action on $\mathbb{C}[z]/z^p$ is given by the q-differential operators*

$$E = \frac{1}{\mathfrak{q} - \mathfrak{q}^{-1}} z(1 - \mathcal{E}^{-1}),$$
$$K = \mathcal{E}^{-1},$$
$$F = \frac{1}{\mathfrak{q} - \mathfrak{q}^{-1}} \partial.$$

Proof. First, by **2.3.1**, E, K, and F are q-differential operators. Next, we verify that the right-hand sides of the three formulas above act on the z^m as desired. This suffices for the proof, but it is actually rather instructive to verify the $\mathcal{U}$ commutation relations for the above E, K, and F. For example, we have

$$\begin{aligned} EF - FE &= \frac{1}{(\mathfrak{q} - \mathfrak{q}^{-1})^2} z(1 - \mathcal{E}^{-1})\partial - \frac{1}{(\mathfrak{q} - \mathfrak{q}^{-1})^2} \partial z(1 - \mathcal{E}^{-1}) \\ &= \frac{1}{(\mathfrak{q} - \mathfrak{q}^{-1})^2} (1 - \mathfrak{q}^{-2}\mathcal{E}^{-1}) z\partial - \frac{1}{(\mathfrak{q} - \mathfrak{q}^{-1})^2} \partial z(1 - \mathcal{E}^{-1}) = \frac{\mathcal{E}^{-1} - \mathcal{E}}{\mathfrak{q} - \mathfrak{q}^{-1}}, \end{aligned}$$

where in the last equality we substitute $z\partial = \mathfrak{q}(1 - \mathcal{E})$ and $\partial z = \mathfrak{q} - \mathfrak{q}^{-1}\mathcal{E}$. □

2.4. Decomposition of $\overline{\mathbb{C}}_{\mathfrak{q}}[z,\partial]$. We now decompose the p^2-dimensional $\mathcal{U}$-module $\overline{\mathbb{C}}_{\mathfrak{q}}[z,\partial]$ into indecomposable representations.

2.4.1. $\mathcal{P}_1^+$. The projective module $\mathcal{P}_1^+ \subset \overline{\mathbb{C}}_{\mathfrak{q}}[z,\partial]$ is identified very easily. For t in (1.8), it follows that

$$Et = z + \mathfrak{q} z^p \partial^{p-1}, \qquad Ft = \partial + \mathfrak{q} z^{p-1} \partial^p.$$

In $\overline{\mathbb{C}}_{\mathfrak{q}}[z,\partial]$, we therefore have the $\mathcal{P}_1^+$ module realized as shown in (1.7) (where, again, the horizontal arrows represent the action of F and E up to nonzero factors).

2.4.2. Theorem. *As a $\mathcal{U}$-module, $\overline{\mathbb{C}}_{\mathfrak{q}}[z,\partial]$ decomposes as*

$$\overline{\mathbb{C}}_{\mathfrak{q}}[z,\partial] = \mathcal{P}_1^+ \oplus \mathcal{P}_3^+ \oplus \dots \oplus \mathcal{P}_\nu^+,$$

where $\nu = p$ if p is odd and $p-1$ if p is even.

(We recall that $\dim\mathcal{P}^+_n = 2p$ for $1 \leqslant n \leqslant p-1$ and $\dim\mathcal{P}^+_p = p$.)

Proof. The proof is only half legerdemain and the other half calculation, somewhat involved at one point; reducing the calculational component would be desirable.

The module $\mathcal{P}^+_1$ is given in (1.7). The module $\mathcal{P}^+_p$, which occurs in the direct sum in the theorem whenever $p = 2s+1$ is odd, is the irreducible representation with the highest-weight vector

$$t_1(s) = \sum_{i=0}^{s} \mathfrak{q}^{is} \begin{bmatrix} s+i-1 \\ i \end{bmatrix} z^{i+s}\partial^i, \qquad p = 2s+1.$$

Calculating with the aid of

$$E(z^m\partial^n) = \mathfrak{q}^{1-n}[n]z^m\partial^{n-1} - \mathfrak{q}^{m-2n}[m]z^{m+1}\partial^n,$$
$$F(z^m\partial^n) = \mathfrak{q}^{1-m}[m]z^{m-1}\partial^n - \mathfrak{q}^{n-2m}[n]z^m\partial^{n+1},$$

we easily verify that $Et_1(s) = 0$; it also follows that $F^{p-1}t_1(s) \neq 0$; in fact,

$$F^{p-1}t_1(s) = [p-1]!\sum_{i=0}^{s} \mathfrak{q}^{is} \begin{bmatrix} s+i-1 \\ i \end{bmatrix} z^i\partial^{i+s}.$$

As we know from [1], each of the other $\mathcal{P}^+_{2r+1}$ modules for $1 \leqslant r \leqslant \lfloor \frac{p-1}{2} \rfloor$ has the structure (with r omitted from arguments for brevity)

(2.1)
$$\begin{tikzcd}[column sep=small]
 & & t_1 \arrow[r, shift left] \arrow[dl, "E"'] & \dots \arrow[l, shift left] \arrow[r, shift left] & t_{2r+1} \arrow[l, shift left] \arrow[dr, "F"] & & \\
l_{p-2r-1} \arrow[r, shift left] & \dots \arrow[l, shift left] \arrow[r, shift left] & l_1 \arrow[l, shift left] \arrow[dr, "F"'] & & & r_1 \arrow[r, shift left] \arrow[dl, "E"] & \dots \arrow[l, shift left] \arrow[r, shift left] & r_{p-2r-1} \arrow[l, shift left] \\
 & & & b_1 \arrow[r, shift left] & \dots \arrow[l, shift left] \arrow[r, shift left] & b_{2r+1} \arrow[l, shift left] & &
\end{tikzcd}$$

and our task is now to identify the corresponding elements in $\overline{\mathbb{C}}_{\mathfrak{q}}[z,\partial]$.

We begin constructing $\mathcal{P}^+_{2r+1}$ from the bottom, setting

$$b_1 = \sum_{i=0}^{p-r-1} \frac{[r+i-1]!}{[i]!}\,\mathfrak{q}^{ri}z^{i+r}\partial^i,$$

which is easily verified to satisfy the relation $Eb_1 = 0$; also, $F^{2r}b_1 \neq 0$ — in fact,

$$F^{2r}b_1 = [2r]!\sum_{i=0}^{p-r-1} \frac{[r+i-1]!}{[i]!}\,\mathfrak{q}^{ri}z^i\partial^{i+r}$$

— and $F^{2r+1}b_1 = 0$. This completely describes the bottom $(2r+1)$-dimensional submodule (the irreducible representation $\mathcal{X}^+_{2r+1}$).

We next seek l_1 such that $b_1 = Fl_1$; obviously, l_1 is of the general form

$$l_1 = \sum_{i=0}^{p-r-2} \lambda_i\mathfrak{q}^{ri}z^{i+r+1}\partial^i.$$

The condition $b_1 = Fl_1$ is equivalent to the recursion relations (we restore r in the argument)

$$\lambda_{i+1}(r)[i+r+2] - \mathfrak{q}^{-2r-1}[i]\lambda_i(r) = \mathfrak{q}^{r+i+1}\frac{[i+r]!}{[i+1]!}. \tag{2.2}$$

The problem is made nontrivial by the existence of *two* boundary conditions: we must have

$$\lambda_0(r) = \mathfrak{q}^r\frac{[r-1]!}{[r+1]} \tag{2.3}$$

and

$$\lambda_{p-r-2}(r) = \mathfrak{q}^{2r}\frac{[r]!}{[r+2]} \tag{2.4}$$

simultaneously.

We now solve the recursion starting from the $i=0$ boundary. The problem is thus to find $\lambda_i(r)$ with $i \geqslant 1$ from (2.2) and (2.3) and then verify that (2.4) is satisfied.

The solution is particularly simple for $r=1$, where $\lambda_i(1) = \mathfrak{q}^2/[3]$ for all $i \geqslant 1$. For $r=2$, the solution is "linear in i":

$$\lambda_i(2) = \begin{bmatrix}5\\2\end{bmatrix}^{-1}(\mathfrak{q}^3[i+4] + \mathfrak{q}^4[i-1]), \quad i \geqslant 1.$$

For $r=3$, it is "quadratic" in a similar sense,

$$\lambda_i(3) = \begin{bmatrix}7\\3\end{bmatrix}^{-1}\Bigl(\mathfrak{q}^4[i+5][i+6] + \mathfrak{q}^5[i+5]\begin{bmatrix}3\\2\end{bmatrix}[i-1] + \mathfrak{q}^6[i-2][i-1]\Bigr), \quad i \geqslant 1.$$

The general solution is given by

$$\begin{aligned}\lambda_i(r) = \begin{bmatrix}2r+1\\r\end{bmatrix}^{-1}\Biggl(&\mathfrak{q}^{r+1}\begin{bmatrix}i+2r\\r-1\end{bmatrix}[r-1]! + \\ &+\sum_{n=2}^{r-1}\mathfrak{q}^{r+n}\begin{bmatrix}i+2r+1-n\\r-n\end{bmatrix}\begin{bmatrix}r-1\\n\end{bmatrix}\begin{bmatrix}r\\n-1\end{bmatrix}[r-n-1]!\prod_{j=1}^{n-1}[i-j] + \\ &+\mathfrak{q}^{2r}\prod_{j=1}^{r-1}[i-j]\Biggr),\end{aligned}$$

$i \geqslant 1$. The first term in the brackets can be included into the sum over n, by extending it to $n=1$, but we isolated it because this is the only term that does not contain the factor $[i-1]$ and it clearly shows that the solution starts as $\left[\begin{smallmatrix}2r+1\\r\end{smallmatrix}\right]^{-1}\mathfrak{q}^{r+1}[i+r+2]\dots[i+2r]$ (all the other terms are then found relatively easily from the recursion). The boundary condition at $i = p-r-2$ is remarkably simple to verify: only one (the last) term contributes and immediately yields the desired result.

The structure of the general formula may be clarified with a more representative example:

$$\lambda_i(5) = \begin{bmatrix}11\\5\end{bmatrix}^{-1}\Bigl(\mathfrak{q}^6[i+10][i+9][i+8][i+7] + \mathfrak{q}^7[i+9][i+8][i+7]\begin{bmatrix}5\\2\end{bmatrix}[i-1] \\ + \mathfrak{q}^8[i+8][i+7]\frac{[4]}{[2]}\begin{bmatrix}5\\2\end{bmatrix}[i-2][i-1] + \mathfrak{q}^9[i+7]\begin{bmatrix}5\\2\end{bmatrix}[i-3][i-2][i-1] \\ + \mathfrak{q}^{10}[i-4][i-3][i-2][i-1]\Bigr).$$

This also illustrates the general situation with the boundary condition at $i = p-r-2$ (only the last term is nonzero in $\lambda_{p-7}(5)$).

With the λ_i and l_1 thus found, the other l_n follow by the action of E.

All the r_n in (2.1), starting with r_1 such that $Er_1 = b_{2r+1}$, are found totally similarly (or, with some care, obtained from the l_n by interchanging z and ∂).

The proof is finished with a recourse to the representation theory of $\mathcal{U}$ [4]. For definiteness, we consider the case of an odd p, $p = 2s+1$. Then what we have established so far is the existence of elements shown with black dots in Fig. 1, for the irreducible projective

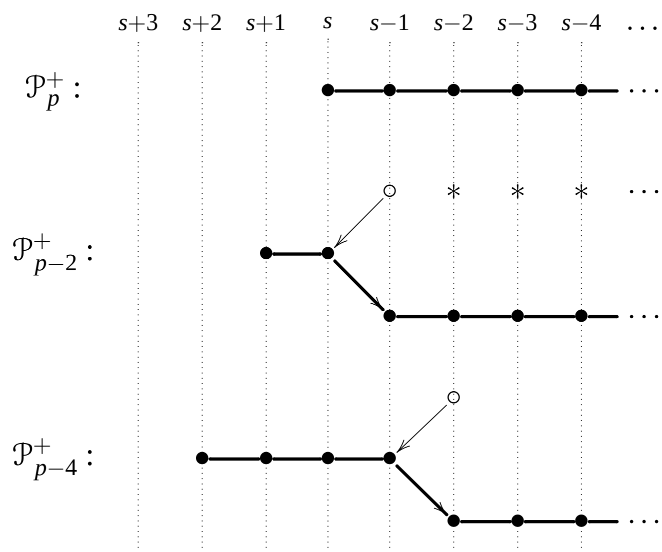

FIGURE 1. Identifying the projective modules in $\overline{\mathbb{C}}_{\mathfrak{q}}[z,\partial]$.

module $\mathcal{P}^+_p$ and for what is to become the projective modules $\mathcal{P}^+_{p-2}$, $\mathcal{P}^+_{p-4}$, $\dots$, $\mathcal{P}^+_1$. To actually show that the black dots do complete to the respective projective modules, we establish the arrows (maps by E) from some elements shown with open dots (which are thus to become the corresponding t_1 in (2.1)). The grading indicated in the figure is such that $\deg z = 1$ and $\deg\partial = -1$. In any grade $u > 0$, there are $p-u$ linearly independent elements in $\overline{\mathbb{C}}_{\mathfrak{q}}[z,\partial]$:

$$z^u, \quad z^{u+1}\partial, \quad z^{u+2}\partial^2, \quad \dots, \quad z^{p-1}\partial^{p-1-u}.$$

In grade s, in particular, there are $p-s=s+1$ elements, and just $s+1$ black dots in all of the $\mathcal{P}^+_p$, $\mathcal{P}^+_{p-2}$, $\dots$, $\mathcal{P}^+_1$. But in grade $s-1$, there are $s+2$ linearly independent elements, only $s+1$ of which have been accounted for by the black dots constructed so far. We let the remaining element — the open dot in grade $s-1$ in Fig. 1 — be temporarily denoted by $\circ_{s-1}$.

Because grade s is exhausted by black dots, $E(\circ_{s-1})$ is either zero or a linear combination of the $\bullet$s. But it is elementary to see that there is only one (up to a nonzero factor, of course) element in each grade annihilated by E, and in grade $s-1$ it has already been found: this is the b_1 state (the leftmost $\bullet$) in $\mathcal{P}^+_{p-2}$ (once again, in what is to become $\mathcal{P}^+_{p-2}$ when we finish the proof). Therefore, $E(\circ_{s-1})$ is a linear combination of the $\bullet$s in grade s, but we know from [4] that this can only be the corresponding element of the $\mathcal{P}^+_{p-2}$ module (the reason is that this is the only element in this grade that is annihilated by F in a quotient of $\overline{\mathbb{C}}_{\mathfrak{q}}[z,\partial]$).

Once the $\bullet\!\swarrow\!\circ$ arrow from a *single* element in grade $s-1$ is thus established, the rest of the $\mathcal{P}^+_{p-2}$ module is completed automatically [4]. In particular, there are the $*$s shown in Fig. 1, and hence just one missing $\overline{\mathbb{C}}_{\mathfrak{q}}[z,\partial]$ element in grade $s-2$, to which we again apply the above argument. Repeating this gives all of the projective modules in (1.4). $\square$

3. MATRIX REPRESENTATION

3.1. The matrix representation of the basic commutation relation (1.5) is found quite straightforwardly (it has many parallels in the q-literature, but nevertheless seems to be new). Because both z and ∂ are p-nilpotent, the matrices representing them have to be triangular and start with a next-to-leading diagonal; Eq. (1.5) then fixes the matrices as in (1.10) (modulo similarity transformations). The rest is just a matter of direct verification (and, of course, a consequence of the fact that $\dim\overline{\mathbb{C}}_{\mathfrak{q}}[z,\partial]=p^2$).

As regards the $\mathcal{U}$ action in the explicit form (1.3), we first verify it on the generators, ∂ and z represented as in (1.10), and then propagate to $\mathrm{Mat}_p(\mathbb{C})$ in accordance with the $\mathcal{U}$-module algebra property.

It is amusing to see how the $\mathcal{U}$-module algebra property $h(XY)=\sum h'(X)h''(Y)$ holds for the ordinary matrix multiplication. For $h=F$, for example, we have (for "bulk" values of i and j)

$$\begin{aligned}\Bigl(\sum F'(X)F''(Y)\Bigr)_{ij} &= \sum_{k=1}^{p}\bigl(K^{-1}(X)\bigr)_{ik}\,(F(Y))_{kj}+\sum_{k=1}^{p}(F(X))_{ik}\,(Y)_{kj}\\ &=\sum_{k=1}^{p-1}\mathfrak{q}^{k-2i+1}x_{ik}[k]y_{k+1,j}-\sum_{k=1}^{p}\mathfrak{q}^{j-2i}[j-1]x_{ik}y_{k,j-1}\end{aligned}$$

$$+\mathfrak{q}^{1-i}[i]x_{i+1,1}y_{1,j}+\sum_{k=1}^{p-1}\bigl(\mathfrak{q}^{1-i}[i]x_{i+1,k+1}-\mathfrak{q}^{k-2i+1}[k]x_{i,k}\bigr)y_{k+1,j}$$

$$=-\sum_{k=1}^{p}\mathfrak{q}^{j-2i}[j-1]x_{ik}y_{k,j-1}+\sum_{k=0}^{p-1}\mathfrak{q}^{1-i}[i]x_{i+1,k+1}y_{k+1,j},$$

which is $(F(XY))_{ij}$. The formulas for $E(XY)$ are equally straightforward.

3.2. Examples.

3.2.1. As another example of "matrices as a visual aid," we note that the cointegral $\mathbf{\Lambda}\in\mathcal{U}$ must map any $X\in\mathrm{Mat}_p(\mathbb{C})$ into the unit matrix times a factor; with the cointegral normalized as in [1],

$$\mathbf{\Lambda}=\sqrt{\frac{p}{2}}\,\frac{1}{([p-1]!)^2}F^{p-1}E^{p-1}\sum_{j=0}^{2p-1}K^j,$$

we actually have

$$\mathbf{\Lambda}(X)=\mathbf{1}\Bigl((-1)^p\sqrt{2p}\sum_{i=1}^{p}\mathfrak{q}^{2i-1}x_{ii}\Bigr).$$

Also, it is easy to see that in the matrix form, the b_1 (bottom left) element of each $\mathcal{P}^+_{2r+1}$ ($r\geqslant 1$) is the one-diagonal lower-diagonal matrix

$$(b_1(r))_{ij}=\delta_{i,j+r}\,\mathfrak{q}^{2r(j-1)}[r-1]!\,.$$

3.2.2. We choose the "moderately large" value $p=4$ for further illustration. Then the idea of how the $\mathcal{U}$ generators act on the matrices is clearly seen from

$$(\mathfrak{q}-\mathfrak{q}^{-1})EX=\begin{pmatrix}x_{12}&x_{13}&x_{14}&0\\-x_{11}+x_{22}&\mathfrak{q}^2x_{12}+x_{23}&x_{13}+x_{24}&-\mathfrak{q}^2x_{14}\\-\mathfrak{q}^2x_{21}+x_{32}&-x_{22}+x_{33}&\mathfrak{q}^2x_{23}+x_{34}&x_{24}\\x_{31}+x_{42}&-\mathfrak{q}^2x_{32}+x_{43}&-x_{33}+x_{44}&\mathfrak{q}^2x_{34}\end{pmatrix},$$

$$(\mathfrak{q}-\mathfrak{q}^{-1})^2E^2X=$$

$$\begin{pmatrix}x_{13}&x_{14}&0&0\\(\mathfrak{q}^2-1)x_{12}+x_{23}&(\mathfrak{q}^2+1)x_{13}+x_{24}&(1-\mathfrak{q}^2)x_{14}&0\\\mathfrak{q}^2x_{11}-(\mathfrak{q}^2+1)x_{22}+x_{33}&-\mathfrak{q}^2x_{12}+(\mathfrak{q}^2-1)x_{23}+x_{34}&\mathfrak{q}^2x_{13}+(\mathfrak{q}^2+1)x_{24}&-\mathfrak{q}^2x_{14}\\-\mathfrak{q}^2x_{21}+(1-\mathfrak{q}^2)x_{32}+x_{43}&\mathfrak{q}^2x_{22}-(\mathfrak{q}^2+1)x_{33}+x_{44}&(\mathfrak{q}^2-1)x_{34}-\mathfrak{q}^2x_{23}&\mathfrak{q}^2x_{24}\end{pmatrix},$$

$$(\mathfrak{q}-\mathfrak{q}^{-1})^3E^3X=\begin{pmatrix}x_{14}&0&0&0\\\mathfrak{q}^2x_{13}+x_{24}&x_{14}&0&0\\x_{12}-x_{23}+x_{34}&\mathfrak{q}^2x_{24}-x_{13}&x_{14}&0\\\mathfrak{q}^2x_{11}-\mathfrak{q}^2x_{33}-x_{22}+x_{44}&-x_{12}+x_{23}-x_{34}&-\mathfrak{q}^2x_{13}-x_{24}&x_{14}\end{pmatrix},$$

and

$$FX = \begin{pmatrix} x_{21} & x_{22} - x_{11} & (-\mathfrak{q}^2 - 1)x_{12} + x_{23} & x_{24} - \mathfrak{q}^2 x_{13} \\ (1 - \mathfrak{q}^2)x_{31} & \mathfrak{q}^2 x_{21} + (1 - \mathfrak{q}^2)x_{32} & (\mathfrak{q}^2 - 1)x_{22} + (1 - \mathfrak{q}^2)x_{33} & (1 - \mathfrak{q}^2)x_{34} - x_{23} \\ -\mathfrak{q}^2 x_{41} & x_{31} - \mathfrak{q}^2 x_{42} & (\mathfrak{q}^2 + 1)x_{32} - \mathfrak{q}^2 x_{43} & \mathfrak{q}^2 x_{33} - \mathfrak{q}^2 x_{44} \\ 0 & -\mathfrak{q}^2 x_{41} & (1 - \mathfrak{q}^2)x_{42} & x_{43} \end{pmatrix}.$$

4. Differential calculus on $\Omega\overline{\mathbb{C}}_{\mathfrak{q}}[z,\partial]$

We construct a quantum de Rham complex $(\Omega\overline{\mathbb{C}}_{\mathfrak{q}}[z,\partial], d)$ of $\overline{\mathbb{C}}_{\mathfrak{q}}[z,\partial]$ where the differential d commutes with the $\mathcal{U}$ action. This requires introducing a somewhat unusual (compared to the quantum plane case [28, 38]) action of $\mathcal{U}$ on the differentials $dz \equiv \zeta$ and $d\partial \equiv \delta$.

4.1. Let $\mathbb{C}_{\mathfrak{q}}[\zeta,\delta]$ be the unital algebra with the relations

$$\begin{aligned} \zeta^2 = 0, \quad \delta^2 = 0, \\ \delta\,\zeta = -\mathfrak{q}^{-2}\zeta\,\delta. \end{aligned} \tag{4.1}$$

On $\mathbb{C}_{\mathfrak{q}}[z,\partial]\otimes\mathbb{C}_{\mathfrak{q}}[\zeta,\delta]$, we define the differential as

$$d(z) = \zeta, \quad d(\partial) = \delta, \quad d(\zeta) = 0, \quad d(\delta) = 0 \tag{4.2}$$

(and $d(1) = 0$) and set

$$\begin{aligned} \zeta\,z = \mathfrak{q}^{-2}z\,\zeta, \qquad \delta\,\partial = \mathfrak{q}^2\partial\,\delta, \\ \zeta\,\partial = \mathfrak{q}^2\partial\,\zeta, \qquad \delta\,z = \mathfrak{q}^{-2}z\,\delta. \end{aligned} \tag{4.3}$$

The first line here immediately implies that

$$d(z^m) = \mathfrak{q}^{1-m}[m]z^{m-1}\zeta, \qquad d(\partial^n) = \mathfrak{q}^{n-1}[n]\partial^{n-1}\delta.$$

4.1.1. Lemma. *The algebra on z, ∂, ζ, and δ with relations* (1.5), (4.1), *and* (4.3) *and differential* (4.2) *is an associative differential algebra.*

The proof is by direct verification.[7]

[7] As regards comparison with the more familiar case of the Wess–Zumino differential calculus on the quantum plane [28, 38], it may be interesting to note that the associativity requires the vanishing of *both* coefficients ν and β in the tentative relations $\zeta\,\partial = \mu\,\partial\,\zeta + \nu\,z\,\delta$ and $\delta\,z = \alpha\,z\,\delta + \beta\,\partial\,\zeta$. However, similarities with the quantum plane, genuine of superficial, come to an end when we consider the quantum group action: the formulas in **4.2** bear little resemblance to the quantum plane case.

4.2. We next define a $\mathcal{U}$ action on the above algebra by setting

$$E\zeta = -[2]z\zeta, \qquad K\zeta = \mathfrak{q}^2\zeta, \qquad F\zeta = 0,$$
$$E\delta = 0, \qquad K\delta = \mathfrak{q}^{-2}\delta, \qquad F\delta = -\mathfrak{q}^2[2]\partial\,\delta.$$

4.2.1. Lemma. *This defines a differential $\mathcal{U}$-module algebra.*

The proof amounts to verifying that this action preserves the two-sided ideal generated by (4.1)–(4.3).

4.2.2. We note simple consequences of the above formulas:

$$E^i(z^m\,\zeta) = (-1)^i\mathfrak{q}^{im+\frac{i(i-1)}{2}}\begin{bmatrix}m+i+1\\ i\end{bmatrix}[i]!\,z^{m+i}\,\zeta,$$
$$F^i(z^m\,\zeta) = \mathfrak{q}^{i(1-m)+\frac{i(i-1)}{2}}\begin{bmatrix}m\\ m-i\end{bmatrix}[i]!\,z^{m-i}\,\zeta,$$
$$E^i(\partial^m\,\delta) = \mathfrak{q}^{-i(m+1)+\frac{i(i-1)}{2}}\begin{bmatrix}m\\ m-i\end{bmatrix}[i]!\,\partial^{m-i}\,\delta,$$
$$F^i(\partial^m\,\delta) = (-1)^i\mathfrak{q}^{i(m+2)+\frac{i(i-1)}{2}}\begin{bmatrix}m+i+1\\ i\end{bmatrix}[i]!\,\partial^{m+i}\,\delta.$$

In particular,

$$E(z^m\,\zeta) = -\mathfrak{q}^m[m+2]\,z^{m+1}\,\zeta,$$
$$F(\partial^m\,\delta) = -\mathfrak{q}^{m+2}[m+2]\,\partial^{m+1}\,\delta.$$

4.3. Because $d(z^p) = 0$ and $d(\partial^p) = 0$, it follows that the differential $\mathcal{U}$-module algebra structure descends to the quotient by the relations $z^p = 0$ and $\partial^p = 0$. We finally let $\Omega\overline{\mathbb{C}}_{\mathfrak{q}}[z,\partial]$ denote the resulting differential $\mathcal{U}$-module algebra — the sought quantum de Rham complex

$$\Omega\overline{\mathbb{C}}_{\mathfrak{q}}[z,\partial] = \big(\overline{\mathbb{C}}_{\mathfrak{q}}[z,\partial]\otimes\mathbb{C}_{\mathfrak{q}}[\zeta,\delta],d\big)\big/\mathscr{I},$$

where $\mathscr{I}$ is the ideal generated by (1.5), (1.6), and (4.1)–(4.3).

As a vector space, $\Omega\overline{\mathbb{C}}_{\mathfrak{q}}[z,\partial]$ naturally decomposes into zero-, one- and two-forms. In $\Omega^1\overline{\mathbb{C}}_{\mathfrak{q}}[z,\partial]$, the elements $z^{p-1}\,\zeta$ and $\partial^{p-1}\,\delta$ are the cohomology of d (the "cohomology corners" of the modules shown in (1.12)).

5. Conclusions

As noted above, it is a classic result that (using the modern nomenclature) the matrix algebra is generated by the generators x and y of a finite quantum plane (with $x^p = y^p = 1$) at the corresponding root of unity [2]; it may be even better known that the quantum plane carries a quantum-$s\ell(2)$ action [28, 38]; and the two facts can of course be combined to produce a quantum-$s\ell(2)$ action on matrices (cf. [23, 39]). We construct an action of

$\overline{\mathcal{U}}_{\mathfrak{q}} s\ell(2)$ at $\mathfrak{q} = e^{\frac{i\pi}{p}}$ on $p \times p$ matrices starting not from the quantum plane but from q-differential operators on a "quantum line"; the explicit formulas for this action are not altogether unworthy of consideration.

Also, the $\overline{\mathcal{U}}_{\mathfrak{q}} s\ell(2)$-module algebra constructed here (and most "invariantly" described in terms of q-differential operators) is relevant in view of the Kazhdan–Lusztig correspondence between $\overline{\mathcal{U}}_{\mathfrak{q}} s\ell(2)$ and the $(p,1)$ logarithmic conformal models. Previously, the Kazhdan–Lusztig correspondence in logarithmic conformal field theories has been observed to hold at the level of representation theories (of the quantum group and of the chiral algebra) and modular transformations (on the quantum group center and on generalized characters of the chiral algebra) [1, 4, 5, 6, 7]. Our results show how it can be extended to the level of fields, the key observation being that the object required on the quantum-group side is an algebra with "good" properties under the action of $\mathcal{U}$ and with a differential that commutes with this action.

Another possibility to look at the Kazhdan–Lusztig correspondence is offered just by the $\overline{\mathcal{U}}_{\mathfrak{q}} s\ell(2)$-module algebra defined on $\mathrm{Mat}_p(\mathbb{C})$: a "spin chain" can be defined by placing the algebra generated by z and ∂ at each site (as we remember, these generalize free fermions, which indeed occur at $p = 2$). In choosing the Hamiltonian, an obvious option is to have it related to the Virasoro generator L_0; a suggestive starting point on a finite lattice is the relation [4]

$$e^{2i\pi L_0} = \boldsymbol{v},$$

where $\boldsymbol{v}$ is the ribbon element in $\overline{\mathcal{U}}_{\mathfrak{q}} s\ell(2)$. In the matrix language, the spin chain with the $\overline{\mathcal{U}}_{\mathfrak{q}} s\ell(2)$-module algebra generated by z and ∂ at each site is equivalently described just by letting $\overline{\mathcal{U}}_{\mathfrak{q}} s\ell(2)$ act on $\mathrm{Mat}_p(\mathbb{C}) \otimes \mathrm{Mat}_p(\mathbb{C}) \otimes \dots$, which may be helpful in practical computations. (This construction may have some additional interest because the relevant action is nonsemisimple (cf. [40, 41, 42, 43]), but at the same time the indecomposable representations occurring here are under control due to the decomposition in (1.4).) In addition, it is also interesting to answer several questions "on the $\overline{\mathbb{C}}_{\mathfrak{q}}[z, \partial]$ side," such as where the even-dimensional modules $\mathcal{X}^{+}_{2r}$ and their projective covers $\mathcal{P}^{+}_{2r}$ are hiding.

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APPENDIX A. OPE ALGEBRAS AND PARAFERMIONIC STATISTICS

We outline how the parafermionic statistics can be incorporated into conformal field theory.

A.1. Background: OPE. For conformal fields (operators) $A(w)$, $B(w)$, $\dots$ defined on the complex plane, the purpose of the OPE algebra [44, 45][8] is to calculate the expressions (referred to as OPE poles) $[A,B]_n$ in "short-distance expansions"

$$A(z)\,B(w) = \sum_{n\ll\infty} \frac{[A,B]_n(w)}{(z-w)^n} \tag{A.1}$$

for any composite operators A and B in terms of the $[\ ,\ ]_m$ specified for a set of "basis" operators. (By a composite operator of any $A(w)$ and $B(w)$, we mean $[A,B]_0(w)$, which is also called the normal-ordered product and is often written as $AB(w)$ or $A(w)B(w)$.) The rules for calculating the OPEs are [44, 45]

$$[B,A]_n = (-1)^{AB} \sum_{\ell\geqslant n} \frac{(-1)^\ell}{(\ell-n)!} d^{\ell-n}[A,B]_\ell,$$

$$[A,[B,C]_0]_n = (-1)^{AB}[B,[A,C]_n]_0 + \sum_{\ell=0}^{n-1} \binom{n-1}{\ell} [[A,B]_{n-\ell},C]_\ell,$$

where in the sign factor $(-1)^{AB}$ — the signature of the Fermi statistics — A and B denote the Grassmann parities of the corresponding operators.[9]

The first of the above rules allows computing the "transposed" OPE $B(z)\,A(w)$ once the OPE $A(z)B(w)$ is known; the second rule is the prescription for calculating an OPE with a composite operator $[B,C]_0$. There is a third rule stating that d acts on the normal-ordered product $[A,B]_0$ as derivation. These three rules (and the simple relation $[dA,B]_n = -(n-1)[A,B]_{n-1}$) suffice for the calculation of any OPE of composite operators [45].

Each of the two formulas above inevitably contains an inversion of the operator order (accompanied by a sign factor for fermions); this is where a generalization to the parafermionic statistics is to be made.

A.2. Parafermionic OPE. We assume that the fields carry a quantum group action and that an R-matrix is given. As a generalized "transposition" OPE rule, we then postulate

$$[B,A]_k = \sum_{\ell\geqslant k} \frac{(-1)^\ell}{(\ell-k)!} d^{\ell-k}[R^{(2)}(A),R^{(1)}(B)]_\ell, \tag{A.2}$$

where $R^{(2)}$ and $R^{(1)}$ are understood just as in (1.9) (Sweedler's summation is implied), and where we assume that all the OPEs in the right-hand side are known. For the "composite"

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[8] We proceed in rather down-to-earth terms; see [46] and the references therein for a much more elaborate approach.

[9] And d is the operator of differentiation with respect to the coordinate on the complex plane; we use this notation instead of the more common ∂ so as not to add to the notation overload already existing with "z," which is now a coordinate on the complex plane along with w.

OPE rule, similarly, we set

$$[A,[B,C]_0]_k = [R^{(2)}(B),[R^{(1)}(A),C]_k]_0 + \sum_{\ell=0}^{k-1}\binom{k-1}{\ell}[[A,B]_{k-\ell},C]_\ell. \tag{A.3}$$

The consistency of these formulas is not obvious a priori, already because of the new fields, except B and A themselves, occurring under the action of the "right and left coefficients" of the R-matrix, in $R^{(2)}(B)$ and $R^{(1)}(A)$. In general, moreover, whenever a transposition of two fields does not square to the identity transformation (the situation generally referred to as "fractional statistics"), some cuts on the complex plane must be chosen (or a cover of the complex plane should be specified on which the fields are defined). Furthermore, the proposed OPE rules should also be extended to include possible occurrences of $\log(z-w)$, which we leave for future work. But it is interesting to see how the scheme may work for our R-matrix (1.17) and "parafermionic" fields modeled on the projective module in (1.7).

A.3. The $\overline{\mathcal{U}}_{\mathfrak{q}}s\ell(2)$ example. We introduce $p-1$ pairs of conformal fields $\zeta^m(w)$ and $\delta^m(w)$, $m=1,\dots,p-1$, carrying the same $\mathcal{U}$ action as the z^m and ∂^m in Sec. 2, i.e.,

$$\begin{aligned}
&E^i\zeta^m(w) = (-1)^i\mathfrak{q}^{im+\frac{i(i-1)}{2}}\begin{bmatrix}i+m-1\\m-1\end{bmatrix}[i]!\,\zeta^{m+i}(w),\\
&F^i\zeta^m(w) = \mathfrak{q}^{i(1-m)+\frac{i(i-1)}{2}}\begin{bmatrix}m\\m-i\end{bmatrix}[i]!\,\zeta^{m-i}(w),
\end{aligned}\qquad K\zeta^m(w)=\mathfrak{q}^{2m}\zeta^m(w),$$

$$\begin{aligned}
&E^i\delta^m(w) = \mathfrak{q}^{i(1-m)+\frac{i(i-1)}{2}}\begin{bmatrix}m\\m-i\end{bmatrix}[i]!\,\delta^{m-i}(w),\\
&F^i\delta^m(w) = (-1)^i\mathfrak{q}^{im+\frac{i(i-1)}{2}}\begin{bmatrix}i+m-1\\m-1\end{bmatrix}[i]!\,\delta^{m+i}(w),
\end{aligned}\qquad K\delta^m(w)=\mathfrak{q}^{-2m}\delta^m(w),$$

with $\delta^0(w)=\zeta^0(w)=1$ (and, formally, $\delta^m(w)=\zeta^m(w)=0$ for $m<0$ or $m\geqslant p$). Here, $w\in\mathbb{C}$, which is our "space–time."

We also have the derivative of each field, $d\zeta^m(w)$ and $d\delta^m(w)$, which we view as space–time 1-forms, and hence regard d as a differential. The differential must commute with the quantum group action, just as the differential d in Sec. 4, which allows the algebraic constructions involving the differential to be carried over to the fields.

To summarize the notational correspondence between Secs. 2–4 and this Appendix, we write the dictionary

$$\begin{aligned}
&z^m|_{\text{Sec. 2}}\leftrightarrow\zeta^m(w)|_{\text{App}}, && \partial^m|_{\text{Sec. 2}}\leftrightarrow\delta^m(w)|_{\text{App}}, && m=0,\dots,p-1,\\
&d(z^m)|_{\text{Sec. 2}}\leftrightarrow d\zeta^m(w)|_{\text{App}}, && d(\partial^m)|_{\text{Sec. 2}}\leftrightarrow d\delta^m(w)|_{\text{App}}, && m=1,\dots,p-1
\end{aligned}\tag{A.4}$$

(we recall that $\zeta^0(w)=\delta^0(w)=1$), or, using (1.14),

$$\partial^{m-1}\,\delta|_{\text{Sec. 4}}\leftrightarrow\eta^m(w)|_{\text{App}},\qquad m=1,\dots,p-1.$$

A.3.1. Either E or F (depending on the conventions) is to be associated with the action of a screening operator in conformal field theory (cf. [1]); screenings commute with Virasoro generators and therefore do not change the conformal weight. Because we have the maps $F:\zeta^1(w)\to 1$ and $E:\delta^1(w)\to 1$, it follows that both $\delta^n(w)$ and $\zeta^n(w)$ must have conformal weight 0 (see (1.13)).

We then fix the basic OPEs of weight-0 fields:

$$\delta^m(z)\,\zeta^n(w)=[m]\delta^{m,n}\log(z-w).$$

Nonlogarithmic OPEs occur when the derivative of either $\zeta^n(w)$ or $\delta^n(z)$ is taken:

$$d\delta^m(z)\,\zeta^n(w)=\frac{[m]\delta^{m,n}}{z-w},\qquad \delta^m(z)\,d\zeta^n(w)=-\frac{[m]\delta^{m,n}}{z-w}.$$

A.3.2. As we have noted, fractional-statistics fields generally require cuts on the complex plane, because taking one of such fields around another is not an identity transformation. Therefore, for each ordered pair of fields (A,B), we must specify whether formula (A.2) is to be used with R or R^{-1}. The rule that we adopt in the current case can be formulated in terms of diagrams of type (1.13): we do *not* use the formulas with the R-matrix when both $R^{(1)}$ and $R^{(2)}$ act toward the socle (the bottom submodule) in (1.13).

For example, this rule allows rewriting Λ with the reversed normal-ordered products as

$$\Lambda=\sum_{n=1}^{p-1}\frac{1}{[n]}\,[R^{(2)}(\delta^n),R^{(1)}(\zeta^n)]_0=\sum_{n=1}^{p-1}\sum_{i=0}^{p-1}\frac{g(i,n)}{[n]}[\delta^{n+i},\zeta^{n+i}]_0=\sum_{n=1}^{p-1}\frac{\mathfrak{q}^{-2n}}{[n]}\,[\delta^n,\zeta^n]_0, \tag{A.5}$$

where both $R^{(2)}\sim F^i$ and $R^{(1)}\sim E^i$ act "to the outside," and where we use the temporary notation

$$g(i,n)=(\mathfrak{q}-\mathfrak{q}^{-1})^i\mathfrak{q}^{\frac{i(i-1)}{2}-i^2-i-2n(i+n)}\begin{bmatrix}i+n-1\\n-1\end{bmatrix}^2[i]!\,.$$

The same strategy yields the transposed OPE $\zeta^n(z)\,d\delta^m(w)$:

$$[\zeta^m,d\delta^n]_1=-[R^{(2)}(d\delta^n),R^{(1)}(\zeta^m)]_1=-\delta^{m,n}\sum_{i=0}^{p-1}g(i,n)[n+i]=-\delta^{m,n}\mathfrak{q}^{2n}[n],$$

or, in a human-friendly form,

$$\zeta^m(z)\,d\delta^n(w)=-\frac{\delta^{m,n}\mathfrak{q}^{2n}[n]}{z-w}.$$

Thus, the effect of the R-matrix reduces in these cases to the phase factor $\mathfrak{q}^{2n}=e^{\frac{2i\pi n}{p}}$ occurring under transposition.

A.3.3. As a further example, we use the elementary OPEs just obtained to calculate

$$[d\zeta^m,\Lambda]_1 = \sum_{n=1}^{p-1}\frac{1}{[n]}[R^{(2)}(\zeta^n),[R^{(1)}(d\zeta^m),\delta^n]_1]_0$$
$$= \sum_{i=0}^{p-1}(\mathfrak{q}-\mathfrak{q}^{-1})^i(-1)^i\mathfrak{q}^{\frac{i(i-1)}{2}+2m(i+m)}\begin{bmatrix}m+i\\ m\end{bmatrix}\begin{bmatrix}i+m-1\\ m-1\end{bmatrix}[i]!\,\mathfrak{q}^{2(m+i)}\zeta^m = \zeta^m.$$

It then follows that $[\Lambda,d\zeta^m]_1 = -[R^{(2)}(d\zeta^m),R^{(1)}(\Lambda)]_1 = -[d\zeta^m,\Lambda]_1 = -\zeta^m$ because only the $i=0$ term in the R-matrix contributes to $[\ ,\ ]_1$.

Next, trying to directly apply (A.3) to calculate $[d\delta^m,\Lambda]_1$ as

$$[d\delta^m,\Lambda]_1 = \sum_{n=1}^{p-1}\frac{1}{[n]}\Big([R^{(2)}(\zeta^n),[R^{(1)}(d\delta^m),\delta^n]_1]_0 + [[d\delta^m,\zeta^n]_1,\delta^n]_0\Big),$$

we encounter the forbidden arrangement of maps by the left and right R-matrix coefficients; anticipating the result, we claim that this is irrelevant in this case (essentially because d in $d\delta^m$ annihilates the submodule spanned by unity), but it is instructive to avoid the forbidden arrangement by using the "reversed" Λ in (A.5):

$$[d\delta^m,\Lambda]_1 = \sum_{n=1}^{p-1}\frac{\mathfrak{q}^{-2n}}{[n]}[d\delta^m,[\delta^n,\zeta^n]_0]_1 = \sum_{n=1}^{p-1}\frac{\mathfrak{q}^{-2n}}{[n]}[R^{(2)}\delta^n,[R^{(1)}d\delta^m,\zeta^n]_1]_0$$
$$= \sum_{i=0}^{m-1}(\mathfrak{q}-\mathfrak{q}^{-1})^i\mathfrak{q}^{\frac{i(i-1)}{2}+2m(m-i)}(-1)^i\begin{bmatrix}m-1\\ m-i-1\end{bmatrix}\begin{bmatrix}m\\ m-i\end{bmatrix}[i]!\,\delta^m = \delta^m.$$

It also follows that $[\Lambda,d\delta^m]_1 = -\delta^m$.

A.3.4. A "parafermionic" $\zeta\eta$ system. Returning to the OPEs in **A.3.2**, we represent the derivative of $\delta^n(w)$ as in (1.14). Then the fields $\zeta^n(w)$ and $\eta^n(w)$, whose OPEs are given by

$$\eta^m(z)\,\zeta^n(w) = \frac{\delta^{m,n}\mathfrak{q}^{1-n}}{z-w}, \qquad \zeta^m(z)\eta^n(w) = -\frac{\delta^{m,n}\mathfrak{q}^{n+1}}{z-w},$$

make up a $(p-1)$-component "parafermionic" first-order system; it generalizes the free fermions, which are indeed recovered for $p=2$, when also $m=n=1$ (and $\mathfrak{q}=i$). The behavior of the $\eta^n(w)$ under the $\mathcal{U}$ action is given by the formulas in **4.2.2**, in accordance with the dictionary in (A.4).

Similarly to the case of free fermions, we have the weight-1 field (a current) $\mathcal{J} = \sum_{n=1}^{p-1}\mathfrak{q}^{n-1}[\zeta^n,\eta^n]_0$. From (1.12), we conclude that it participates in the diagram

(A.6)
$$\begin{tikzcd}[column sep=small]
\mathcal{J}(w) = \sum_{n=1}^{p-1}\mathfrak{q}^{n-1}\zeta^n\eta^n(w) \arrow[dr, "F"] & & & & ? \arrow[dl, "E"'] \\
& \eta^1(w) \arrow[r, shift left] & \dots \arrow[l, shift left] \arrow[r, shift left] & \eta^{p-1}(w), \arrow[l, shift left] &
\end{tikzcd}$$

where it remains to identify the "cohomology corner" *in terms of fields* (we do not have an $\eta^p(w)$, see (A.4)).

The "corner" must be a field of the same conformal weight as the current $\mathcal{J}(w)$, but must not be a bilinear combination of the $\zeta^m(w)$ and $\eta^m(w)$. It is naturally provided by the setting in [15], where the chiral algebra $W(p)$ and its representation spaces are defined as the kernel of the "short" screening operator S_-, whereas the "long" screening S_+ acts on the fields. The action of a screening S amounts to taking the first-order pole in the OPE with the screening current $s(w)$, which is often expressed as

$$S_\pm = \oint s_\pm(w)$$

(with a contour integration over w implied). In the standard realization in terms of a free bosonic field $\phi(w)$, we have $s_+(w) = e^{\sqrt{2p}\,\phi(w)}$ and $s_-(w) = e^{-\sqrt{2/p}\,\phi(w)}$. With $E \in \mathcal{U}$ identified with the screening operator S_-, we now rescale the grading used in Fig. 1 as follows: $\mathcal{J}(w)$ in (A.6) is assigned degree 0 and each F arrow increases the degree by $\sqrt{2/p}$. Then the question mark in (A.6) has the degree $\sqrt{2p}$, and therefore *the cohomology corner is filled with the screening current* $s_+(w)$. We thus obtain (1.15).

A field realization of the other module in (1.12) requires taking a "dual" picture, in terms of the first-order "parafermionic" system comprised by the $d\zeta^m(w)$ and $\delta^m(w)$ and the $\mathcal{J}(w)$ current used to construct the screening.[10]

A.3.5. For the current $\mathcal{J}(w)$, the rules in **A.2** lead to the standard OPE $[\eta^m, \mathcal{J}]_1 = \eta^m$. Transposing, we then find $[\mathcal{J}, \eta^m]_1 = -[R^{(2)}(\eta^m), R^{(1)}(\mathcal{J})]_1 = -[\eta^m, \mathcal{J}]_1 = -\eta^m$ because only the $i = 0$ term in R-matrix (1.17) contributes. Although $\mathcal{J}(w)$ is not a $\mathcal{U}$-invariant, it behaves like one in a number of OPEs.

We next calculate the first (and the only) pole in the OPE $\zeta^m(z)\mathcal{J}(w)$:

$$\begin{aligned}
[\zeta^m, \mathcal{J}]_1 &= \sum_{n=1}^{p-1} \mathfrak{q}^{n-1}\,[R^{(2)}(\zeta^n), [R^{(1)}(\zeta^m), \eta^n]_1]_0\\
&= -\sum_{i=0}^{p-1} (\mathfrak{q} - \mathfrak{q}^{-1})^i \mathfrak{q}^{\frac{i(i-1)}{2} + 2(m+i)m} (-1)^i \begin{bmatrix} i+m-1\\ m-1 \end{bmatrix} \begin{bmatrix} m+i\\ m \end{bmatrix} [i]!\, \mathfrak{q}^{2(m+i)}\, \zeta^m = -\zeta^m.
\end{aligned}$$

It now readily follows that $[\mathcal{J}, \zeta^m]_1 = \zeta^m$.

[10]None of these free-field systems, as is well known from the $p = 2$ example, allows constructing "logarithmic" modules of the Virasoro or triplet algebra, i.e., indecomposable modules where L_0 is not diagonalizable. Logarithmic modules require an integration, such as $d^{-1}\eta^n(w)$, leading to the ζ^n, δ^n fields. A remarkable trace of this integration may already be observed at the algebraic level in (1.8) — the q-integers in the denominator and an "integration constant" $\alpha 1$.

An instructive calculation is that of the $\mathcal{J}(z)\,\mathcal{J}(w)$ OPE:

$$[\mathcal{J},\mathcal{J}]_2 = \sum_{n=1}^{p-1} \mathfrak{q}^{n-1}[[\mathcal{J},\zeta^n]_1,\eta^n]_1 = \sum_{n=1}^{p-1} \mathfrak{q}^{n-1}[\zeta^n,\eta^n]_1 = -\sum_{n=1}^{p-1} \mathfrak{q}^{2n} = 1.$$

Thus, although $\mathcal{J}(w)$ is a sum of the $p-1$ terms $\mathfrak{q}^{n-1}\,\zeta^n\eta^n(w)$, it does *not* show the factor $p-1$ in the $\mathcal{J}(z)\,\mathcal{J}(w)$ OPE.

Naturally, just the same is observed in the "dual" description, in terms of another first-order system, with the current $\mathcal{I}$ in (1.16). With the OPEs $[\delta^m,\mathcal{I}]_1 = -\delta^m$ and $[d\zeta^m,\mathcal{I}]_1 = d\zeta^m$ (where in the last formula the calculation is very much that for $[d\zeta^m,\Lambda]_1$), it follows that $[\mathcal{I},\mathcal{I}]_2 = -\sum_{n=1}^{p-1}\mathfrak{q}^{2n} = 1$, just as for the $\mathcal{J}$ current.

A.3.6. The same "summation to minus unity" occurs for the simplest energy–momentum tensor, the normal ordered product

$$\mathcal{T} = \sum_{n=1}^{p-1} \frac{1}{[n]}\,[d\zeta^n, d\delta^n]_0 = \sum_{n=1}^{p-1} \mathfrak{q}^{n-1}\,[d\zeta^n,\eta^n]_0.$$

It is a $\mathcal{U}$ invariant, which reduces the OPE calculations to the standard, except at the last step in calculating half the central charge:

$$[\mathcal{T},\mathcal{T}]_4 = \sum_{n=1}^{p-1} \mathfrak{q}^{n-1}\Bigl(3[d\zeta^n,\eta^n]_2 + [d^2\zeta^n,\eta^n]_3\Bigr) = (3-2)\sum_{n=1}^{p-1}\mathfrak{q}^{2n} = -1$$

and, similarly,

$$[\mathcal{T},\mathcal{J}]_3 = -1.$$

The energy-momentum tensor can of course be "improved" by the derivative of a current. The "$\mathcal{J}$-improved" energy–momentum tensor

$$\tilde{\mathcal{T}} = \mathcal{T} - \beta\, d\mathcal{J}$$

has the central charge $-2-12\beta^2+12\beta$, which coincides with the one of the $(p,1)$ model for

$$\beta = \bigl(1+\tfrac{1}{\sqrt{2p}}\bigr)\bigl(1-\sqrt{\tfrac{p}{2}}\bigr).$$

References

[1] B.L. Feigin, A.M. Gainutdinov, A.M. Semikhatov, and I.Yu. Tipunin, *Modular group representations and fusion in logarithmic conformal field theories and in the quantum group center*, Commun. Math. Phys. 265 (2006) 47–93 [hep-th/0504093].

[2] H. Weyl, *The Theory of Groups and Quantum Mechanics*, Dover Publications (1931).

[3] M. Cohen and S. Westreich, *From supersymmetry to quantum commutativity*, J. Algebra 168 (1994) 1–27.

[4] B.L. Feigin, A.M. Gainutdinov, A.M. Semikhatov, and I.Yu. Tipunin, *Kazhdan–Lusztig correspondence for the representation category of the triplet W-algebra in logarithmic CFT*, Theor. Math. Phys. 148 (2006) 1210–1235 [math.QA/0512621].

[5] B.L. Feigin, A.M. Gainutdinov, A.M. Semikhatov, and I.Yu. Tipunin, *Logarithmic extensions of minimal models: characters and modular transformations*, Nucl. Phys. B 757 (2006) 303–343 [hep-th/0606196].

[6] B.L. Feigin, A.M. Gainutdinov, A.M. Semikhatov, and I.Yu. Tipunin, *Kazhdan–Lusztig-dual quantum group for logarithmic extensions of Virasoro minimal models*, J. Math. Phys. 48 (2007) 032303 [math.QA/0606506].

[7] A.M. Semikhatov, *Factorizable ribbon quantum groups in logarithmic conformal field theories*, Theor. Math. Phys. 154 (2008) 433–453 [arXiv:0705.4267 [hep-th]].

[8] A.M. Gainutdinov, *A generalization of the Verlinde formula in logarithmic CFT*, Teor. Mat. Fiz. 159 No. 2 (2009) 193–205.

[9] H.G. Kausch, *Extended conformal algebras generated by a multiplet of primary fields*, Phys. Lett. B 259 (1991) 448.

[10] M.R. Gaberdiel and H.G. Kausch, *Indecomposable fusion products*, Nucl. Phys. B477 (1996) 293–318 [hep-th/9604026].

[11] M.R. Gaberdiel and H.G. Kausch, *A rational logarithmic conformal field theory*, Phys. Lett. B 386 (1996) 131–137 [hep-th/9606050].

[12] M.R. Gaberdiel, *An algebraic approach to logarithmic conformal field theory*, Int. J. Mod. Phys. A18 (2003) 4593–4638 [hep-th/0111260].

[13] M. Flohr, *Bits and pieces in logarithmic conformal field theory*, Int. J. Mod. Phys. A18 (2003) 4497–4592 [hep-th/0111228].

[14] J. Fjelstad, J. Fuchs, S. Hwang, A.M. Semikhatov, and I.Yu. Tipunin, *Logarithmic conformal field theories via logarithmic deformations*, Nucl. Phys. B633 (2002) 379–413 [hep-th/0201091].

[15] J. Fuchs, S. Hwang, A.M. Semikhatov, and I.Yu. Tipunin, *Nonsemisimple fusion algebras and the Verlinde formula*, Commun. Math. Phys. 247 (2004) 713–742 [hep-th/0306274].

[16] M. Flohr and H. Knuth, *On Verlinde-like formulas in $c_{p,1}$ logarithmic conformal field Theories*, arXiv:0705.0545 [math-ph].

[17] D. Adamović and A. Milas, *On the triplet vertex algebra $W(p)$*, Adv. Math. 217 (2008) 2664–2699 [arXiv:0707.1857v2 [math.QA]].

[18] M. Flohr amd M.R. Gaberdiel, *Logarithmic torus amplitudes*, J. Phys. A39 (2006) 1955–1968 [hep-th/0509075].

[19] M.R. Gaberdiel and I. Runkel, *The logarithmic triplet theory with boundary*, J. Phys. A39 (2006) 14745–14780 [hep-th/0608184].

[20] M.R. Gaberdiel and I. Runkel, *From boundary to bulk in logarithmic CFT*, J. Phys. A41 (2008) 075402 [arXiv:0707.0388 [hep-th]].

[21] A. Alekseev, D. Gluschenkov, and A. Lyakhovskaya, *Regular representation of the quantum group $sl_q(2)$ (q is a root of unity)*, St. Petersburg Math. J. 6 (1994) 88.

[22] R. Coquereaux, *On the finite dimensional quantum group $M_3 \oplus (M_{2|1}(\Lambda^2))_0$*, Lett. Math. Phys. 42 (1997) 309–328 [hep-th/9610114].

[23] L. Dąbrowski, F. Nesti, and P. Siniscalco, *A finite quantum symmetry of $M(3,\mathbb{C})$*, hep-th/9705204.

[24] R. Coquereaux, A.O. García, and R. Trinchero, *Differential calculus and connections on a quantum plane at a cubic root of unity*, Rev. Math. Phys. 12 (2000) 227 [math-ph/9807012].

[25] Y. Arike, *Symmetric linear functions of the restricted quantum group $\overline{U}_q sl_2(\mathbb{C})$*, arXiv:0706.1113.

[26] P. Furlan, L. Hadjiivanov, and I. Todorov, *Zero modes' fusion ring and braid group representations for the extended chiral WZNW model*, arXiv:0710.1063.

[27] D. Kazhdan and G. Lusztig, *Tensor structures arising from affine Lie algebras,* I, J. Amer. Math. Soc. 6 (1993) 905–947; II, J. Amer. Math. Soc. 6 (1993) 949–1011; III, J. Amer. Math. Soc. 7 (1994) 335–381; IV, J. Amer. Math. Soc. 7 (1994) 383–453.

[28] J. Wess and B. Zumino, *Covariant differential calculus on the quantum hyperplane*, Nucl. Phys. B (Proc. Suppl.) 18 (1990) 302–312.

[29] A. Giaquinto and J. Zhang, *Quantizations and deformations of Weyl algebras*, J. Alg. 176 (1995) 861–881.

[30] N. Jing and J. Zhang, *Quantum Weyl algebras and deformations of $U(G)$*, Pac. J. Math. 171 (1995) 437–454.

[31] H.G. Kausch, *Symplectic fermions*, Nucl. Phys. B583 (2000) 513–541 [hep-th/0003029].

[32] A.M. Semikhatov, *Higher string functions, higher-level Appell functions, and the logarithmic $\widehat{s\ell}(2)/u(1)$ conformal field theory model*, Commun. Math. Phys. 286 (2009) 559–592 [arXiv:0710.2028 [math.QA]].

[33] F.A. Smirnov, *Quantum groups and generalized statistics in integrable models*, Commun. Math. Phys. 132 (1990) 415–439.

[34] J.-H. Lu, *Hopf algebroids and quantum groupoids*, Int. J. Math., 7 (1996) 47–70 [math.QA/9505024].

[35] T. Brzeziński and G. Militaru, *Bialgebroids, $\times_A$-bialgebras and duality*, J. Algebra 251 (2002) 279–294 [math.QA/0012164].

[36] L.A. Lambe and D.E. Radford, *Algebraic aspects of the quantum Yang–Baxter equation*, J. Algebra 154 (1993) 228–288.

[37] K. Erdmann, E.L. Green, N. Snashall, and R. Taillefer, *Representation theory of the Drinfeld doubles of a family of Hopf algebras,* J. Pure and Applied Algebra 204 (2006) 413–454 [math.RT/0410017].

[38] Yu. I. Manin, *Notes on quantum groups and quantum de Rham complexes*, Teor. Mat. Fiz. 92 (1992) 425–450.

[39] R. Coquereaux and G.E. Schieber, *Action of a finite quantum group on the algebra of complex $N \times N$ matrices,* AIP Conf. Proc. 453 (1998) 9–23 [math-ph/9807016].

[40] H. Hinrichsen and V. Rittenberg, *Quantum groups, correlation functions and infrared divergences*, Phys. Lett. B304 (1993) 115–120 [hep-th/9301056].

[41] A. Nichols, V. Rittenberg, and J. de Gier, *One-boundary Temperley–Lieb algebras in the XXZ and loop models*, J. Stat. Mech. 0503 (2005) 003 [cond-mat/0411512].

[42] P.A. Pearce, J. Rasmussen, and J.-B. Zuber, *Logarithmic Minimal Models*, J. Stat. Mech. 0611 (2006) P017 [hep-th/0607232]

[43] N. Read and H. Saleur, *Enlarged symmetry algebras of spin chains, loop models, and S-matrices*, Nucl. Phys. B 777 (2007) 263 [cond-mat/0701259].

[44] F.A. Bais, P. Bouwknegt, M. Surridge, and K. Schoutens, *Extensions of the Virasoro algebra constructed from Kac–Moody algebras using higher order Casimir invariants,* Nucl. Phys. B 304 (1988) 348–370.

[45] K. Thielemans, *A Mathematica package for computing operator product expansions*, Int. J. Mod. Phys. C 2 (1991) 787.

[46] M. Rosellen, *OPE-Algebras and their modules*, math.QA/0312313.

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